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Calculus

Md. Shouvik Iqbal

Part I

The Basis of Modern Calculus—focuses on the introduction of limits, properties of limits, analytic evaluation of limits, limits involving infinity, continuity, formal def. of limits

Part II

Differential Calculus—focuses on the introduction of differential calculus, derivation of the derivative of functions, applications of derivatives

Part III

Integral Calculus—focuses on the introduction of integral calculus, connection between differential and integral calculus, methods of integration, applications of integration

Calculus

Md. Shouvik Iqbal

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A Message from the Author

Over the course of three years, more than 1,000 hours have been meticulously dedicated to authoring, researching, editing, reviewing, designing, illustrating, typesetting, formatting, indexing, proofreading, troubleshooting, and supervising every other aspect of this book's creation by a one-man army—the author—with the hope that it will provide help to someone in need, that even being entirely free of any cost whatsoever. If this book proves beneficial to you in any way at any point, and if circumstances allow, a generous donation to support this one-man army would be greatly appreciated. You may do so at the link: https://md-shouvik-iqbal.github.io/calculus or by scanning the following QR code.

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Preface

"The only constant in life is change"

Indeed! Everything in life is changing except for the state of being changed itself. From the smallest particles to the largest of galaxies and beyond, everything is in a state of flux—a continuous change. Calculus—as we know it today—is the mathematics of this very change. More precisely, it is the mathematics to deal with variables that are continuously changing and can thus be used to understand the changing nature around us and to answer the fundamental questions within our minds.

Unfortunately, in today's world, the fundamental questions and ideas behind calculus have tragically been reduced to not much more than an intimidating academic obstacle. The subject of deep exploration, requiring years and years to fully comprehend, has now been relegated to the status of a mere requirement—a box to be ticked. The emphasis is no longer put on understanding the deeper meaning behind the ideas; instead, it seems like it has turned to rote memorization of formulas and techniques to reach the correct answer, all in the hope of getting a good score, seen as currency for future opportunities. Should one fail to do so within a certain time period, their prospects—at least in society's eyes—are in jeopardy!

Calculus—at least, in my eyes—is not just all about that. I view it as a profound inspiration where philosophical ideas intersect with that of mathematical reasoning to produce something nothing short of elegant. What inspired this book was my desire to share this elegance I found within calculus. The origin

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of which can be traced back to the collection of notes I made during my own journey of learning the subject years ago. This, I believe, ensures that the topics that fascinated me as a learner will resonate just as deeply with readers. I've made every effort to present the material in a clear and accessible language, without sacrificing the eloquence of tone. Likewise, I tried my best to explain the reasoning behind each concept without excluding any contents covered in any other formal textbook. That is, this book covers all the topics typically included in any introductory calculus text, but it distinguishes itself in how those ideas are explained. Therefore, the reader may find it comfortable to use it as a main text, especially when learning the concepts.

The Structure of the Book.

The book is divided into three parts to discuss two major branches of calculus: differential calculus and integral calculus. The first part serves as a foundation for the contents that follow and consists of the opening chapter. The second part focuses on differential calculus and includes chapters two, three, and four. The third part focuses on integral calculus, covering chapters five through eight. A detailed purpose and direction of each chapter is discussed below.

- **Chapter 1.** The Concept of Limits and Continuity.

  The purpose of this chapter is to introduce the foundational concept of limit and continuity of a function in a way that demonstrates its essence. This chapter is organized in a way that covers all the necessary information about limits and continuity at a foundational level. The chapter begins with a thought-provoking observation and analysis to set the reader in the correct mindset to understand the topics that follow. After giving the reader an informal definition of the limit of a function, the chapter begins to discuss and define what is referred to as one-sided limits and the conditions for a limit to exist. Following that, the chapter dives into a long discussion of evaluating the limit of a function—showing various methods and techniques. Then the limits involving infinity are discussed in a similarly detailed fashion. Following that, the concept of continuity of a function is discussed along with their associated properties. At the end, the chapter concludes by discussing the formal $(\varepsilon - \delta)$ definition of limits along with its execution.

- **Chapter 2.** An Introduction to Differential Calculus.

  The purpose of this chapter, as the name suggests, is to introduce a subfield of calculus—differential calculus. Understanding the ideas covered in this chapter is essential for comprehending the chapters that follow. The chapter begins with a brief overview of differential calculus, leading to a key question related to the problem of the tangent line, which is central to understanding the concept of the derivative. The chapter then discusses the tangent line problem in detail—defining the slope of the tangent line and the instantaneous rate of change. From this, the concept of the derivative is introduced, defined in terms of the slope of the tangent line. After discussing the derivative of a function with several examples, the chapter discusses various notations used to represent derivatives. Finally, the chapter concludes by addressing two closely related concepts: differentiability and continuity.

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- **Chapter 3.** The Derivation of Derivatives.

  The purpose of this chapter is to derive the derivatives of various functions in order to avoid reapplying the definition of the derivative repeatedly. The chapter begins with the section "Elementary Rules of Differentiation," which covers several fundamental rules: Constant Rule, Constant Multiple Rule, Power Rule, Summation Rule, Difference Rule, Product Rule, and Quotient Rule. These rules serve as the foundation for differentiating a wide range of functions. The next section, "Differentiation Rules for Trigonometric Functions," focuses on the derivatives of the six trigonometric functions along with their examples. Following this, the section "Differentiation of Composite Functions" introduces the Chain Rule and explores its application through numerous examples. Understanding the Chain Rule is crucial for comprehending the following topics. Next, in the section "Implicit Differentiation," the process of differentiating implicitly defined functions is discussed, along with multiple examples to illustrate the method. The section "Differentiation Rules for Exponential Functions" derives and discusses the derivatives of exponential functions with various examples. The subsequent section "Differentiation Rules for Logarithmic Functions," derives and discusses the derivatives of logarithmic functions with various examples. Following this, the section on "Logarithmic Differentiation" explains a technique for simplifying functions using logarithmic properties before differentiating, which proves beneficial in various cases. The next section, "Differentiation Rules for Inverse Trigonometric Functions," derives the derivatives of the six inverse trigonometric functions and provides various examples. Following that, the section "Differentiation of General Inverse Functions," extends the discussion to cover general inverse functions. Finally, the chapter concludes with a summary of the key derivatives for quick reference.

- **Chapter 4.** Applications of Derivatives.

  The purpose of this chapter is to show how the concept of derivative of a function, as discussed previously, can be applied to solve a variety of mathematical problems. The chapter begins with the section "Extreme Values of a Function," which covers topics such as absolute (global) extreme values, relative (local) extreme values, critical points, and the location of absolute (global) extreme values. The next section, "Increasing and Decreasing Functions," discusses the functions that are increasing or decreasing, followed by an introduction to the First Derivative Test for determining intervals of increase or decrease. In the section "Concavity and the Second Derivative Test," the chapter discusses the Point of Inflection and the Second Derivative Test as a tool to analyze the concavity of functions and identify points where the concavity changes. Next, the section "Rolle's Theorem and The Mean Value Theorem" presents two important results in differential calculus: Rolle's Theorem and the Mean Value Theorem (MVT), along with their applications. Following that, the section "Related Rates" discusses problems of related rates and shows how the derivatives of multiple variables can be used to solve practical problems with various examples. In "Linearization and Differentials," the concept of linearization is introduced to show how functions can be approximated near a given point using derivatives and is followed by a discussion of differentials. Then the section "Newton's Method" introduces a numerical method to approximate the roots of a function using derivatives. At the end, the chapter concludes with "L'Hôpital's Rule," which provides a method for evaluating limits of indeterminate forms using derivatives.

- **Chapter 5.** An Introduction to Integral Calculus.

  The purpose of this chapter is to introduce a subfield of calculus—Integral calculus. It begins with a discussion of the concept of area and its historical methods of calculation, leading into a discussion of the section "area under a curve." The topics discussed in that section include the summation notation, area approximation, generalization of area approximation, Riemann sums, and the formal definition of the area under a curve in terms of Riemann sums. Following that, the chapter discusses the net signed area in terms of Riemann sums, and introduces the definite integral as a formal way of representing net

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signed area. The properties of definite integrals are then discussed through various examples. Finally, the chapter concludes with a discussion of numerical integration, specifically the Midpoint Rule and the Trapezoidal Rule, which are methods for approximating definite integrals.

- **Chapter 6.** The Connection Between Differential and Integral Calculus.

  The purpose of this chapter is to establish the connection between two subfields of calculus through the Fundamental Theorem of Calculus (FTC). The goal is to show that differentiation and integration are essentially the inverse operations of each other in a way that one can be undone by applying the other. To set up this inverse relationship, the chapter discusses the fundamental topics such as antiderivatives and indefinite integrals and then introduces the Fundamental Theorem of Calculus. Finally, the chapter concludes with a discussion of the inverse relationship between differentiation and integration.

- **Chapter 7.** Methods of Integration.

  The purpose of this chapter is to discuss various methods of integration. It begins with the section "Integration by Substitution," which covers the u-substitution rule through various examples for both definite and indefinite integrals. Next is the section "Integration by Parts," which discusses the integration of products of functions and introduces the LIATE rule, integration by parts for definite integrals, and reduction formulas. Following this, the "Trigonometric Integrals" section discusses methods for integrating functions involving the powers of sine, cosine, tangent, and cotangent functions, as well as the products of these functions. After that comes the "Trigonometric Substitution" section, which discusses techniques for integrating functions of a specific form with the help of trigonometry. The chapter then moves on to "Partial Fractions" to discuss integrating rational functions. Next, the section on "Improper Integrals" discusses integrating functions that either become infinite within an interval or have intervals extending to infinity. Finally, the chapter concludes with a comprehensive table of indefinite integrals as a reference.

- **Chapter 8.** Applications of Integration.

  The purpose of this chapter is to show how integration can be used to solve a range of mathematical problems. It starts with the section "Average Value of a Function" which introduces the concept of average value of a function and the mean value theorem for integrals with examples. The next section "Area of a Region Between Two Curves" explains how to find the area of a region enclosed by two curves with examples. The section "Volume: The Slicing Method" discusses how to find the volume of a three-dimensional solid of revolution. It begins by defining the volume of such solids using integrals and then discusses the disk and washer methods with examples. The following section "Volume: The Method of Cylindrical Shells" presents another method to calculate the volume of a solid of revolution with examples. The section "Arc Length" explains how to precisely calculate the length of a curve using the formula for arc length. The section "The Area of a Surface of Revolution" shows how to calculate the area of the surface of solids of revolution with examples. Finally, the chapter concludes with the section "Work" which discusses the work done by a variable force with examples.

Prerequisite.

The true study of calculus necessitates a certain level of mathematical knowledge, as well as some awareness of its historical context. Although this book is not filled with just mathematically rigorous statements and proofs, it does, however, assume that the readers have a basic understanding of the precalculus topics such as algebra, geometry, and trigonometry. This is because these topics

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are not covered in detail within this book, except when necessary for clarity. While prior exposure to introductory calculus concepts, such as limits, may be beneficial, it is not required in this text. What is required is a solid grasp of the precalculus contents to follow along comfortably.

From my experience, there are three category of potential readers: (1) those who have a strong grasp of the precalculus topics, (2) those with a basic understanding of the precalculus topics, and (3) those with little to no knowledge of the precalculus topics and may struggle comprehending statements such as "Let a be a real number defined over the domain of $f(x)$" or "for all x, there exist y such that this and that is true" or something along those lines.

Readers in the first category can proceed through the book without additional resources. Those in the second category are encouraged to keep a relevant mathematical textbook that covers the precalculus topics or use online resources as needed. However, the third category, lacking a foundation in precalculus, may find the book difficult to follow and are generally advised not to begin without first mastering the necessary background, though this choice is ultimately up to them.

So, the bottom line is that a firm command of precalculus topics is crucial to fully engage with the material, as these topics are not explicitly covered in the book.

About the Author.

Born and raised in Bangladesh, I have completed both my secondary (SSC) and higher secondary (HSC) education under the Board of Intermediate and Secondary Education in Bangladesh. I am currently looking forward to my undergraduate studies. This book began as a collection of notes when I was 16, shortly after my SSC exam and before my admission to intermediate college, and is now being published at the age of 19. Writing this book has been a central focus throughout my academic years. In addition to this, I have a strong interest in academic research in mathematics (as of now, mainly number theory) and philosophy and have authored several research articles and notes. I also have a longstanding engagement with creative disciplines such as painting, music, and literature, for which I have earned awards at both national and international

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levels.

—Md. Shouvik Iqbal

Dhaka, Bangladesh
30 October 2024

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About the Cover

The figure in the cover of this book was taken from the following photograph of the statue of Issac Newton, located in the chapel of Trinity College, Cambridge.

Figure 1: Statue of Isaac Newton

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Part I

The Basis of Modern Calculus

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Chapter 1

Limits and Continuity

In order to travel a distance and reach its end, one must first have to cover half of that distance. From there, to reach the end, the remaining half must be covered, and then again the remaining half, and again the half of what remains, and the process continues endlessly. Each step brings closer and closer to the endpoint, but never exactly reaches it, as there's always another halfway point to be traveled to reach the endpoint. Thus, the conclusion left to be drawn is that the endpoint can never be exactly reached.

This seemingly absurd observation above appears to have been first made by Zeno of Elea, a pre-Socratic Greek philosopher. It is a paradox suggesting that a moving object can certainly approach the endpoint but never exactly reach it. The underlying idea is similar to that of the mathematical concept referred to as the "limit." The notion of such in mathematics stands as such an ineffable construct that, without its existence, the entire field itself might not have developed in the form we see it today. For example, calculus is a fundamental branch of mathematics, and among the significant topics in calculus lie derivatives and integrals, both of which can be thought of as a unique representation of limits. In this chapter, we aim to focus on exploring limits, their definitions, properties, associated theorems, and tackling problems related to limits. Thus, we start with the introduction of the concept of limit.

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§ 1.1 An Introduction to the Limit

The concept of limit is fundamental in modern calculus as it formalizes the informal idea of a variable "approaching a fixed value," which arises in many contexts. Simply put, a limit provides a precise and rigorous way to describe how a function behaves as the input gets closer to a specific point. It captures not only the function's behavior near a point but also how it behaves when it is arbitrarily close to that point. This allows us to analyze situations where direct calculation either fails or leads to ambiguous results by—informally speaking—considering the value predicted by the surrounding points.

Quite frankly, it is hard—if not totally impractical—to try to understand the concept of a limit without first realizing the idea of approaching a value. It's an informal term frequently used in situations such as the following.

Consider a hypothetical bouncing ball "b" with the following property "**P**."

Property **P**: When b is dropped from a certain height on a surface, it rebounds up to *exactly*[1] half of its previous height.

So, we use the symbol "$b(\mathbf{P})$" to denote a bouncing ball with the property **P**.

Say, we drop the ball $b(\mathbf{P})$ from a height of 1 unit. When it is dropped from that height, it bounces back up 0.5 units on the first bounce, then 0.25 units, 0.125 units, and so on. So, the following question arises naturally,

Question: How many times will the ball bounce before it comes to rest on the ground, if it ever does?

Let us logically think about this problem. The ball was released from a height of 1 unit. So, the initial height of that ball is 1 unit. Then, after each bounce, the new height of the ball is half, i.e., $\frac{1}{2}$ times the previous height due to the property **P**. This follows that,

$$\text{after bounce} = 1, \quad \text{the new height} = \underbrace{1}_{\text{previous height}} \times \frac{1}{2} = \frac{1}{2} \text{ units}$$

[1] with no loss of physical properties such as energy in a way that if dropped from a height of x units, it rebounds back up to $\frac{x}{2}$ units after 1 bounce.

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$$\text{after bounce} = 2, \quad \text{the new height} = \underbrace{\frac{1}{2}}_{\text{previous height}} \times \frac{1}{2} = \frac{1}{2^2} \text{ units}$$

$$\text{after bounce} = 3, \quad \text{the new height} = \underbrace{\left(\frac{1}{2} \times \frac{1}{2}\right)}_{\text{previous height}} \times \frac{1}{2} = \frac{1}{2^3} \text{ units}$$

$$\text{after bounce} = 4, \quad \text{the new height} = \underbrace{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)}_{\text{previous height}} \times \frac{1}{2} = \frac{1}{2^4} \text{ units}$$

$$\vdots$$

$$\text{after bounce} = n, \quad \text{the new height} = \underbrace{\overbrace{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}\right)}^{\text{previous height}} \times \frac{1}{2}}_{n} = \frac{1}{2^n} \text{ units}$$

Notice that as the number of bounces n increases, the height $\frac{1}{2^n}$ tends to approach a value of 0, as shown in the table below.

Table 1.1: Table

n	0	1	2	3	4	5	$\cdots$	∞
$\frac{1}{2^n}$	1	0.5	0.25	0.125	0.0625	0.03125	$\cdots$	0

Therefore, we say,

$$\text{as } n \text{ approaches } \infty, \qquad \frac{1}{2^n} \text{ approaches } 0$$

where, n is the number of bounces and $\frac{1}{2^n}$ is the height of the ball $b(\mathbf{P})$ after each bounce. This can symbolically be written as,

$$\text{as } n \to \infty, \qquad \frac{1}{2^n} \to 0$$

by letting the symbol "$\to$" to mean "approaching," just like the symbol "$=$" is used to mean "equal."

Since, the height $\frac{1}{2^n}$ gets closer and closer to a value 0 as the number of bounces n increases without a bound, therefore, the value 0 can be thought of as the ultimate destination—the *limit*—for the height $\frac{1}{2^n}$ to ultimately reach, as n approaches infinity. We express this idea symbolically as follows,

$$\lim_{n \to \infty} \frac{1}{2^n} = 0$$

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In other words, $\frac{1}{2^n} \to 0$, as $n \to \infty$. Thus, we have found our answer to the question above that—hypothetically—the ball $b(\mathbf{P})$ will increasingly approach the ground to settle down as the number of bounces grows without a bound.

Now, notice in the table above that for each bounce n, there exists a corresponding new height $\frac{1}{2^n}$. Therefore, the table above can be modified into a height function f of each bounce n, that is, $f(n) = \frac{1}{2^n}$. Notice that n in the function $f(n) = \frac{1}{2^n}$ is restricted to be a positive integer as the number of bounces can't be negative or fractional. The function, however, maps these positive integers to real numbers as an output.

It is, by no means, essential to do this to understand the concept of approaching a value. Functions can be defined for inputs beyond positive integers for the concept of limits to be still applicable. To demonstrate, consider the same function but with the variable x instead of n, indicating the function is defined on real numbers, as follows.

$$f(x) = \frac{1}{2^x}$$

Visually, it can be seen that $f(x) = \frac{1}{2^x} \to 0$ as $x \to \infty$ in the following figure.

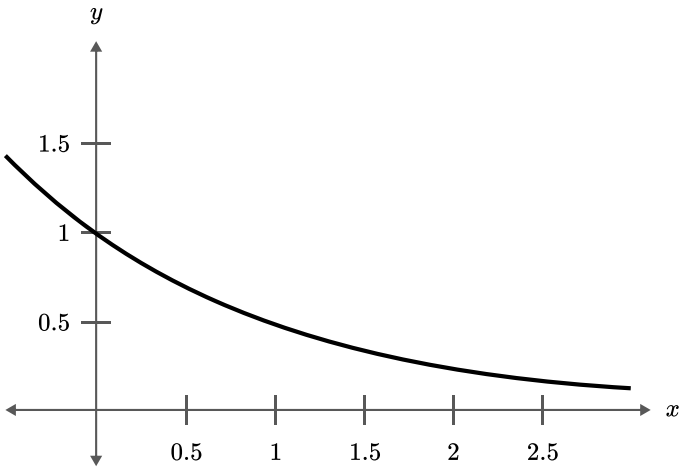

Figure 1.1

Notice that the domain (simply, the input) of $f(n) = \frac{1}{2^n}$ is a subset of the domain of $f(x) = \frac{1}{2^x}$, therefore if the limit of $f(x) = \frac{1}{2^x}$ is 0 as the input approaches infinity, then the limit of $f(n) = \frac{1}{2^n}$ automatically becomes 0 for the

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same.

Now, what if the input x approaches a finite value, say $x = 1$? What value does the function $f(x) = \dfrac{1}{2^x}$ approach? It can be observed—with no effort—from the graph shown below that when $x \to 1$, $f(x) \to 0.5$.

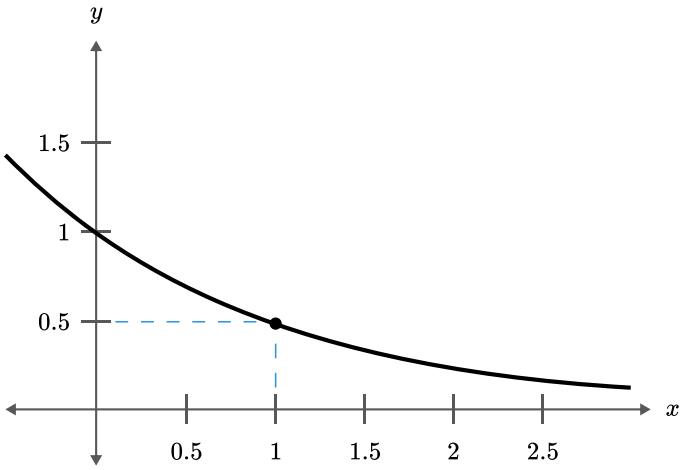

Figure 1.2

It may seem absurd to even consider the concept of approaching a value in this case. Because anyone can evaluate the function $f(x) = \dfrac{1}{2^x}$ at the point $x = 1$ and get the answer that $f(1) = \dfrac{1}{2^1} = 0.5$. However, the idea of limits really becomes necessary when a function is not defined at a point. For example, consider the following function,

$$f(x) = \frac{x^2 - 4^2}{x - 4}$$

What is the value of the function at the point $x = 4$? Evaluating the function at $x = 4$, results in the following,

$$\begin{aligned} f(x) &= \frac{x^2 - 4^2}{x - 4} \\ &= \frac{4^2 - 4^2}{4 - 4} \\ &= \frac{0}{0} \end{aligned}$$

This can be interpreted as the measure of nothing is broken down into the measure of nothing, which is nothing but absurdity. However, by examining the behavior of the function $f(x)$ in the vicinity of $x = 4$, we can predict by

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estimating the value of $f(x)$ at $x = 4$. To do this, we approximate $f(x)$ using values close to $x = 4$, albeit not precisely equal to it. These values may either be slightly less than 4 or slightly greater than 4. We start by estimating values of $f(x)$ using values of x less than 4.

$x < 4$	$f(x) = \dfrac{x^2 - 4^2}{x - 4}$
3.9	7.9
3.99	7.99
3.999	7.999
3.9999	7.9999

The table above indicates that as the values of x are getting closer and closer to the point $x = 4$, the values of $f(x)$ are getting closer and closer to the point $f(x) = 8$. Now, we estimate $f(x)$ using values of x greater than 4.

$x > 4$	$f(x) = \dfrac{x^2 - 4^2}{x - 4}$
4.1	8.1
4.01	8.01
4.001	8.001
4.0001	8.0001

Similar to the previous table, this table above also indicates that as x is approaching 4, $f(x)$ is approaching the value 8.

This can also be seen visually as follows.

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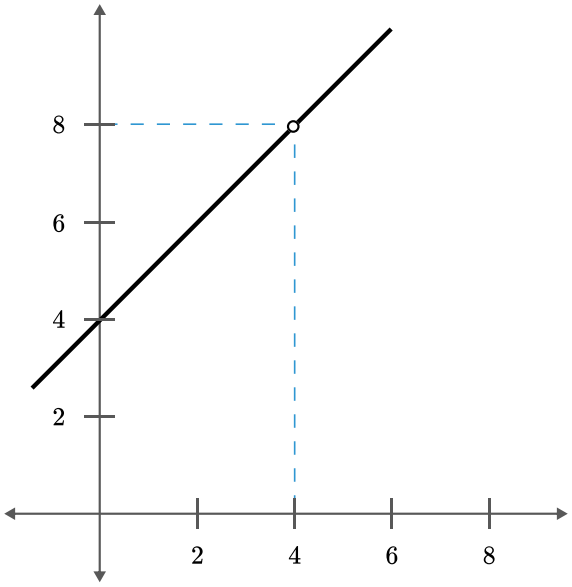

Figure 1.3

Thus, it may be sufficiently logical to conclude that the limit of the function $f(x) = \dfrac{x^2 - 4^2}{x - 4}$, as x gets closer and closer to the value 4, is the value 8.

We can generalize the idea behind the limit and define the limit of a function as follows. Note that this definition is informal and the formal definition is given at the end of this chapter.

Definition 1.1.1. (Informal Definition of Limit)
Let $f(x)$ be a function defined on an open interval containing a, except possibly at the point a. If $f(x)$ gets arbitrarily close to the real number L, as x gets arbitrarily close to the point a, then the limit of $f(x)$ as x approaches a, is L. Symbolically,

$$\lim_{x \to a} f\left(x\right) = L$$

Note that x does not necessarily have to be defined at the point a. In fact, when examining the limit of $f(x)$ as x approaches a, the value of x at a is not even taken into account. The sole concern is how $f(x)$ behaves in the vicinity of a.

By now, if it isn't already obvious, it should be pointed out that as x approaches a particular value, it can approach from two directions on the number line. It can either approach that value through values less than that or through values greater than that particular value, as done previously in the tables above.

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In general terms, if x approaches a finite value a from values less than a, it appears visually to be approaching from the *left side* of a on the number line. If x approaches a from values greater than a, it appears visually to be approaching from the *right side* of a on the number line. Thus, this prompts us to study one-sided limits as outlined in the next section.

———————The End of Section I.I

An Introduction to the Limit

§ 1.2 One-Sided Limits

A one-sided limit refers to the value that a function approaches as the input approaches a particular point from only one side—either from the left or from the right. For example, consider we have an arbitrary function $y = f(x)$, as shown below.

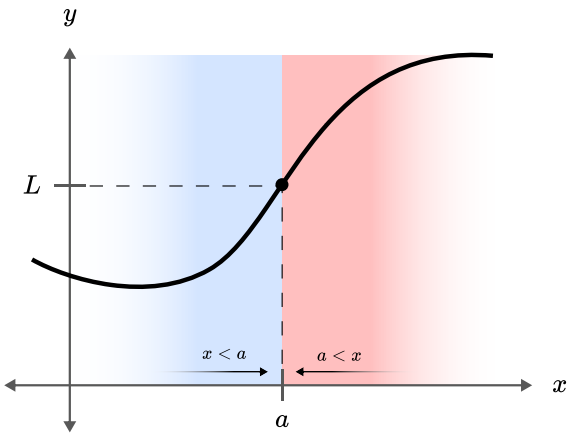

Figure 1.4

Here, as x approaches a, the function $f(x)$ approaches a limit L. If x approaches a through values less than a, it is visually seen that x approaches a from the left side of a on the number line, as indicated by the blue shaded region the figure above. Symbolically, we denote this as $x \to a^-$.

On the other hand, if x approaches a from values greater than a, it is visually seen that x approaches a from the right side of a on the number line, as indicated by the red shaded region in the figure above. Symbolically, this is expressed as $x \to a^+$.

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Thus, the informal definition of this one-sided limit can be given as,

Definition 1.2.1. (One-sided Limits)

1. Limit from the left: Let $f(x)$ be a function defined over an open interval (b, a) and let L be a real number. If $f(x) \to L$ as $x \to a$, where $x < a$, then the limit of $f(x)$ is L as x approaches a from the left. Symbolically,
$$\lim_{x \to a^-} f(x) = L$$
2. Limit from the right: Let $f(x)$ be a function defined over an open interval (a, b) and let L be a real number. If $f(x) \to L$ as $x \to a$, where $a < x$, then the limit of $f(x)$ is L as x approaches a from the right. Symbolically,
$$\lim_{x \to a^+} f(x) = L$$

At first glance, defining one-sided limits might seem trivial. Yet, it turns out to be a crucial clarification. This is because not every function approaches the same limit as its input approaches a single value. For example, consider the graph of the function $f(x) = \dfrac{x}{|x|}$. As $x \to 0$, what value does $y = f(x)$ get closer to?

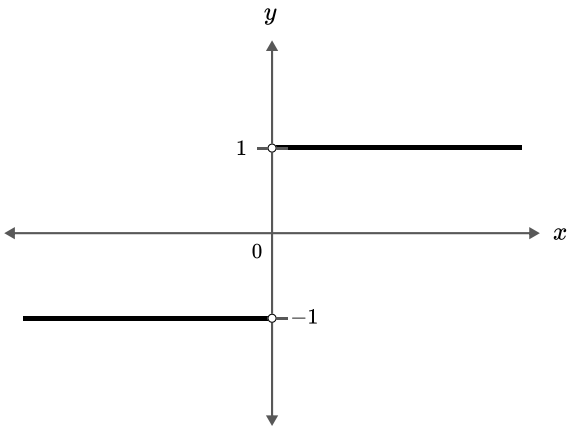

Figure 1.5

The answer is not obvious because if x approaches 0 from the left side of 0, it seems like $f(x)$ approaches the value -1. However, if x approaches 0 from the right side of 0, it seems like $f(x)$ approaches the value 1. Therefore, the overall limit of $f(x)$ as $x \to 0$, does not agree on any single limit and thus does not

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exist, even though side-wise limits do exist, that is,

$$\lim_{x\to 0^-} \frac{x}{|x|} = -1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{x}{|x|} = 1$$

.

Therefore, the situation at hand gives rise to the study of the existence of a limit of a function, as outlined below.

———————The End of Section 1.2

One-Sided Limits

§ 1.3 Existence of a Limit

It follows from the previous section's discussion that a limit L of a function only exists if both the right-sided limit and the left-sided limit yield the same value L. If not, then the limit of the function does not exist, as the function doesn't approach any specific value. Thus, we have the following theorem.

Theorem 1.3.1. (Existence of a Limit) Let $f(x)$ be a function and let a and L be real numbers. Then,

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

This theorem implies,

$$\text{if,} \quad \lim_{x\to a} f(x) \neq L$$

$$\text{then,} \quad \lim_{x\to a^+} f(x) \neq L \quad \text{or,} \quad \lim_{x\to a^-} f(x) \neq L \quad \text{or,} \quad \text{both}$$

While violating Theorem 1.3.1 indeed leads to the absence of a function's limit, it's important to note that this isn't the sole factor causing the existence of a limit to fail. Various other reasons can also contribute to the failure of a function's limit to exist. For example, consider the function below,

$$f(x) = \frac{1}{(x-4)^2}$$

What value does $f(x)$ approach as $x \to 4$? To determine, we look at the graph of $f(x) = \dfrac{1}{(x-4)^2}$, as shown below.

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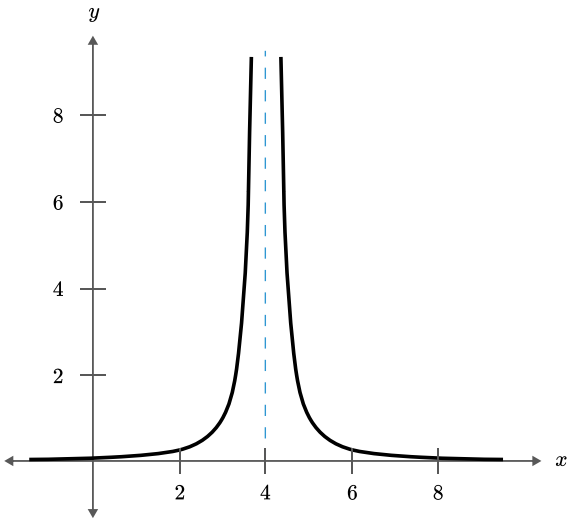

Figure 1.6

It is evident that as $x \to 4$ from both sides, the function $f(x) = \dfrac{1}{(x-4)^2}$ grows with a bound. That is, $f(x) = \dfrac{1}{(x-4)^2} \to \infty$. Since ∞ is not a number, therefore the limit of $f(x) = \dfrac{1}{(x-4)^2}$ as $x \to 4$, does not exists.

One other reason for which the limit of a function fails to exist is oscillating behavior. For example, consider the following function,

$$f(x) = \sin\left(\frac{1}{x}\right)$$

What value does $f(x) = \sin\left(\dfrac{1}{x}\right)$ approach as $x \to 0$?

As $x \to 0$, $\dfrac{1}{x} \to \infty$ and $\sin\left(\dfrac{1}{x}\right)$ oscillate between -1 and 1 infinitely often. Since $f(x) = \sin\left(\dfrac{1}{x}\right)$ does not approach a fixed value, therefore the limit of $f(x) = \sin\left(\dfrac{1}{x}\right)$ as $x \to 0$, does not exist.

In general, the following are the most common reasons for which the limit of a function fails to exist.

Remark 1.3.1.

1. The limit of $f(x)$ differs between the left and right sides, as $x \to a$.

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2. $f(x) \to \pm\infty$, as $x \to a$.
3. $f(x)$ shows oscillating behavior as $x \to a$.

——————The End of Section 1.3

Existence of a Limit

§ 1.4 Analytic Evaluation of Limit of a Function

So far, we have discussed the concept of limits and used two indirect method for their evaluation. The first one is a numerical method, where we determine the limit by numerically analyzing the function near a given point, often by a table of values. The second one is a graphical method, relying on the visual inspection of the graph of a function. However, relying solely on these methods proves inadequate for evaluating the limits of a function. Therefore, in this section, we introduce several theorems regarding limits, enabling the analytical evaluation of the limit of a function.

We begin by discussing a few essential properties of the limit of a function.

Chap. 1 / Sec. 1.4 / Subsec. 1.4.1 : General Properties of Limit of a Function

The following are some of the fundamental properties of the limit of a function, commonly referred to as the "limit laws." By leveraging these laws alongside previously discussed methods, we can evaluate the limit of various kinds of algebraic functions. The proofs for these limit laws are omitted in this section.

Theorem 1.4.1. (Limit Laws)

If L, M, a, and c are real numbers, and

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

then,

1. $\lim\limits_{x\to a} c = c$
2. $\lim\limits_{x\to a} x = a$

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3. $\lim\limits_{x\to a}\Big(c\cdot f\left(x\right)\Big)=c\cdot\lim\limits_{x\to a}f\left(x\right)=c\cdot L$
4. $\lim\limits_{x\to a}\Big(f\left(x\right)+g\left(x\right)\Big)=\lim\limits_{x\to a}f\left(x\right)+\lim\limits_{x\to a}g\left(x\right)=L+M$
5. $\lim\limits_{x\to a}\Big(f\left(x\right)-g\left(x\right)\Big)=\lim\limits_{x\to a}f\left(x\right)-\lim\limits_{x\to a}g\left(x\right)=L-M$
6. $\lim\limits_{x\to a}\Big(f\left(x\right)\cdot g\left(x\right)\Big)=\lim\limits_{x\to a}f\left(x\right)\cdot\lim\limits_{x\to a}g\left(x\right)=L\cdot M$
7. $\lim\limits_{x\to a}\left(\dfrac{f\left(x\right)}{g\left(x\right)}\right)=\dfrac{\lim\limits_{x\to a}f\left(x\right)}{\lim\limits_{x\to a}g\left(x\right)}=\dfrac{L}{M}$ where, $g(x)\neq 0$
8. $\lim\limits_{x\to a}\Big(f\left(x\right)\Big)^{n}=\Big(\lim\limits_{x\to a}f\left(x\right)\Big)^{n}=L^{n}$
9. $\lim\limits_{x\to a}\sqrt[n]{f\left(x\right)}=\sqrt[n]{\lim\limits_{x\to a}f\left(x\right)}=\sqrt[n]{L},$ where, n is a positive integer.

If n is even, then $\lim\limits_{x\to a}f(x)>0$ is assumed.

Acknowledging these limit laws, we now begin to use them to evaluate the limit of a function.

► **Example 1.4.1.** If $f(x)=33$, what value does $f(x)$ approaches as $x\to 2$?

Solution. From Law 1, we know that

$$\lim_{x\to a}c=c$$

This implies,

$$\lim_{x\to 2}33=33$$

Therefore, as $x\to 2$, the function $f(x)\to 33$. ∎

► **Example 1.4.2.** If $f(x)=x$, what value does $f(x)$ approaches as $x\to 2$?

Solution. From Law 2, we know that

$$\lim_{x\to a}x=a$$

This implies,

$$\lim_{x\to 2}x=2$$

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Therefore, as $x \to 2$, the function $f(x) \to 2$. ∎

► **Example 1.4.3.** If $\lim_{x\to 2} f(x) = 4$, then evaluate $\lim_{x\to 2} \pi f(x)$.

Solution. Since π is a constant, therefore by Law 3, we write,

$$\begin{aligned}\lim_{x\to 2} \pi f(x) &= \pi \cdot \lim_{x\to 2} f(x) \\ &= \pi \cdot 4 \\ &= 4\pi\end{aligned}$$

Therefore, the limit of $\pi f(x)$, as $x \to 2$ is 4π. ∎

► **Example 1.4.4.** If $\lim_{x\to e} f(x) = \dfrac{\pi}{2}$ and $\lim_{x\to e} g(x) = 4$, then evaluate

$$\lim_{x\to e} \left(\frac{\pi f(x) + g(x)}{f(x) - f(x)\, g(x)} \right)$$

.

Solution. Since e and π are constants, therefore by the combination of Law 3, 4, 5, 6, and 7, we proceed,

$$\begin{aligned}\lim_{x\to e} \left(\frac{\pi f(x) + g(x)}{f(x) - f(x)\, g(x)} \right) &= \frac{\lim_{x\to e} \Big(\pi f(x) + g(x) \Big)}{\lim_{x\to e} \Big(f(x) - f(x)\, g(x) \Big)} \\ &= \frac{\lim_{x\to e} (\pi \cdot f(x)) + \lim_{x\to e} g(x)}{\lim_{x\to e} f(x) - \lim_{x\to e} (f(x)\, g(x))} \\ &= \frac{\pi \cdot \lim_{x\to e} f(x) + \lim_{x\to e} g(x)}{\lim_{x\to e} f(x) - \lim_{x\to e} f(x) \cdot \lim_{x\to e} g(x)} \\ &= \frac{\pi \cdot \dfrac{\pi}{2} + 4}{\dfrac{\pi}{2} - \dfrac{\pi}{2} \cdot 4} \\ &= \frac{\dfrac{\pi^2 + 8}{2}}{\dfrac{\pi - 4\pi}{2}} \\ &= \frac{\pi^2 + 8}{\cancel{2}} \cdot \frac{\cancel{2}}{-3\pi}\end{aligned}$$

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$$= -\frac{\pi^2 + 8}{3\pi}$$

Therefore, $\lim_{x \to e} \left(\frac{\pi f(x) + g(x)}{f(x) - f(x)\, g(x)} \right) = -\frac{\pi^2 + 8}{3\pi}$, given that $\lim_{x \to e} f(x) = \frac{\pi}{2}$ and $\lim_{x \to e} g(x) = 4$. ∎

► **Example 1.4.5.** If $\lim_{X \to 4} Z(X) = 5$, then evaluate $\lim_{X \to 4} \Big(Z(X) \Big)^5$

Solution. By Law 8, we write,

$$\begin{aligned} \lim_{X \to 4} \Big(Z(X) \Big)^5 &= \Big(\lim_{X \to 4} Z(X) \Big)^5 \\ &= 5^5 \\ &= 3125 \end{aligned}$$

Therefore, the limit of $\lim_{X \to 4} \Big(Z(X) \Big)^5$ is 3125, if $\lim_{X \to 4} Z(X) = 5$. ∎

Chap. 1 / Sec. 1.4 / Subsec. 1.4.2 : Direct Substitution

Direct substitution is one of the simplest methods for evaluating limits. The core idea is to directly substitute the value of x approaching a specific point into the function. If this substitution yields a finite number without any indeterminate forms such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, etc. then the limit is simply that value.

More specifically, for a function $f(x)$ continuous at $x = a$, the limit of $f(x)$ as $x \to a$ is $f(a)$, given that $f(a)$ exists and is finite. That is,

$$\lim_{x \to a} f(x) = f(a)$$

· · · / Subsec. 1.4.2 / Segment 1.4.2.1 : The Limit of Polynomial and Rational Functions

We will now use the limit laws to determine the limit of polynomial functions. For example, take the polynomial function

$$P(x) = x^5 + 2x^4 + 4x^3 + 7x + 9$$

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What value does $P(x)$ approach as $x \to 5$? To determine, we can use the limit laws to evaluate the limit of $P(x)$, as follows.

$$\begin{aligned}
\lim_{x\to 5} P(x) &= \lim_{x\to 5} \left(x^5 + 2x^4 + 4x^3 + 7x + 9\right) \\
&= \lim_{x\to 5} \left(x^5\right) + \lim_{x\to 5} \left(2x^4\right) + \lim_{x\to 5} \left(4x^3\right) + \lim_{x\to 5} (7x) + \lim_{x\to 5} (9) \\
&= \lim_{x\to 5} \left(x^5\right) + 2 \cdot \lim_{x\to 5} \left(x^4\right) + 4 \cdot \lim_{x\to 5} \left(x^3\right) + 7 \cdot \lim_{x\to 5} (x) + \lim_{x\to 5} (9) \\
&= 5^5 + 2 \cdot 5^4 + 4 \cdot 5^3 + 7 \cdot 5 + 9 \\
&= 4919
\end{aligned}$$

From the example above, it can be seen that the limit of the polynomial function $P(x)$ is easily found by direct substitution. That is, if $P(x)$ is a polynomial function, then the limit of it, as $x \to a$, is $P(a)$. This is due to the fact that polynomial functions are one of the simplest types of functions and are defined for all real numbers. Thus, $\lim_{x\to 5} P\left(x\right)$ in the example above, can be evaluated directly as,

$$\begin{aligned}
\lim_{x\to 5} P\left(x\right) &= \lim_{x\to 5} \left(x^5 + 2x^4 + 4x^3 + 7x + 9\right) \\
&= 5^5 + 2 \cdot 5^4 + 4 \cdot 5^3 + 7 \cdot 5 + 9 \\
&= 4919
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 1.4.2. (The Limit of Polynomial Functions) If $P(x)$ is a polynomial function and a is a real number, then,

$$\lim_{x\to a} P(x) = P(a)$$

The theorem above further suggests that if $P(x)$ and $Q(x)$ are polynomial functions such that $Q(x) \neq 0$, then the limit of $\dfrac{P(x)}{Q(x)}$, as $x \to a$, is $\dfrac{P(a)}{Q(a)}$. Thus, we have the following theorem.

Theorem 1.4.3. (The Limit of Rational Functions) If $R(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, and a is a real number such that $Q(x) \neq 0$, then,

$$\lim_{x\to a} R(x) = \lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

A few examples are provided below.

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► **Example 1.4.6.** Evaluate $\lim_{x\to 1} \dfrac{x^3+3x}{x^2+1}$.

Solution. From the theorem above, we write,

$$\begin{aligned}\lim_{x\to 1} \frac{x^3+3x}{x^2+1} &= \frac{1^3+3\cdot 1}{1^2+1}\\ &= \frac{1+3}{1+1}\\ &= \frac{4}{2}\\ &= 2\end{aligned}$$

Therefore, $\lim_{x\to 1} \dfrac{x^3+3x}{x^2+1} = 2$ ∎

► **Example 1.4.7.** If $\dfrac{x^1+x^2+x^3+x^4+x^5}{x^6+x^7+x^8+x^9+x^{10}} \to L$ as $x \to 1$, then what is the value of L?

Solution. From the theorem above, we write,

$$\begin{aligned}\lim_{x\to 1} \left(\frac{x^1+x^2+x^3+x^4+x^5}{x^6+x^7+x^8+x^9+x^{10}}\right) &= \frac{1^1+1^2+1^3+1^4+1^5}{1^6+1^7+1^8+1^9+1^{10}}\\ &= \frac{1+1+1+1+1}{1+1+1+1+1}\\ &= \frac{5}{5}\\ &= 1\end{aligned}$$

Therefore, $\dfrac{x^1+x^2+x^3+x^4+x^5}{x^6+x^7+x^8+x^9+x^{10}} \to 1$ as $x \to 1$. ∎

· · · / Subsec. 1.4.2 / Segment 1.4.2.2 : The Limit of a Reciprocal

The limit of a reciprocal refers to the limit of a function that is the reciprocal of another function as x approaches a certain value. The key idea is to find the limit of $\dfrac{1}{f(x)}$ as $x \to a$. To do this, we need to consider the behavior of $f(x)$ itself. This is summarized below.

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Theorem 1.4.4. (The Limit of a Reciprocal) If $\lim_{x\to a} f(x) = L$ such that $L \neq 0$, then,

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}$$

A few examples are shown below.

► **Example 1.4.8.** Evaluate $\lim_{x\to 2} \frac{1}{2^x}$.

Solution. From the theorem above, we get,

$$\begin{aligned}\lim_{x\to 2} \frac{1}{2^x} &= \frac{1}{2^2} \\ &= \frac{1}{4}\end{aligned}$$

Therefore, $\lim_{x\to 2} \frac{1}{2^x} = \frac{1}{4}$ ∎

► **Example 1.4.9.** If $f(x) = |x^3|$, then what value does $\frac{1}{|x^3|}$ approaches as $x \to -\pi$?

Solution. From the theorem above, we get,

$$\begin{aligned}\lim_{x\to -\pi} \frac{1}{|x^3|} &= \frac{1}{|-\pi^3|} \\ &= \frac{1}{\pi^3}\end{aligned}$$

Therefore, $\frac{1}{|x^3|}$ approaches $\frac{1}{\pi^3}$, as $x \to -\pi$. ∎

· · · / Subsec. 1.4.2 / Segment 1.4.2.3 : The Limit of Transcendental Functions

Transcendental functions are a category of functions that are not algebraic and include functions such as exponential functions, logarithmic functions, trigonometric functions, their inverses, etc. For transcendental functions, the limit of a function as $x \to a$ can be found by directly substituting the value of a into the function, provided that the function is continuous[2] at $x = a$ and it does not result in an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. This is summarized below.

[2] discussed later in this chapter

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Theorem 1.4.5. (The Limit of Transcendental Functions) Let a be a real number defined in the domain of the following functions, then,

1. $\lim_{x\to a} b^x = b^a$,
2. $\lim_{x\to a} \ln(x) = \ln(a)$
3. $\lim_{x\to a} \log_b(x) = \log_b(a)$

Let a be defined in the domain of the following trigonometric functions; then,

1. $\lim_{x\to a} \sin(x) = \sin(a)$
2. $\lim_{x\to a} \cos(x) = \cos(a)$
3. $\lim_{x\to a} \tan(x) = \tan(a)$
4. $\lim_{x\to a} \cot(x) = \cot(a)$
5. $\lim_{x\to a} \sec(x) = \sec(a)$
6. $\lim_{x\to a} \csc(x) = \csc(a)$

Let a be defined in the domain of the following inverse trigonometric functions; then,

1. $\lim_{x\to a} \sin^{-1}(x) = \sin^{-1}(a)$
2. $\lim_{x\to a} \cos^{-1}(x) = \cos^{-1}(a)$
3. $\lim_{x\to a} \tan^{-1}(x) = \tan^{-1}(a)$
4. $\lim_{x\to a} \cot^{-1}(x) = \cot^{-1}(a)$
5. $\lim_{x\to a} \sec^{-1}(x) = \sec^{-1}(a)$
6. $\lim_{x\to a} \csc^{-1}(x) = \csc^{-1}(a)$

A few examples of the theorem above are provided below.

► **Example 1.4.10.** Evaluate $\lim_{x\to 1} e^x$

Solution. From the theorem above, we get,

$$\begin{aligned}\lim_{x\to 1} e^x &= e^1 \\ &= 0\end{aligned}$$

Therefore, $\lim_{x\to 1} e^x = 0$ ∎

► **Example 1.4.11.** Evaluate $\lim_{x\to \pi} \ln(x)$

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Solution. From the theorem above, we get,

$$\lim_{x\to\pi} \ln(x) = \ln(\pi)$$

Therefore, $\lim_{x\to\pi} \ln(x) = \ln(\pi)$. ∎

► **Example 1.4.12.** Evaluate $\lim_{x\to 1} \sin^{-1}(x)$

Solution. From the theorem above, we get,

$$\begin{aligned}\lim_{x\to 1} \sin^{-1}(x) &= \sin^{-1}(1)\\ &= \frac{\pi}{2}\end{aligned}$$

Therefore, $\lim_{x\to 1} \sin^{-1}(x) = \frac{\pi}{2}$. ∎

· · · / Subsec. 1.4.2 / Segment 1.4.2.4 : The Limit of a Composite Function

The limit of a composite function refers to the limit of a function that is the composition of two or more functions. The key idea is that the limit of a composite function can be evaluated by first determining the limit of the inner function and then applying the outer function to that limit. This process relies heavily on the continuity[3] of the outer function f at the limit of the inner function $g(x)$. This is summarized below.

Theorem 1.4.6. (The Limit of a Composite Function) If $f(x)$ and $g(x)$ are functions such that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then,

$$\lim_{x\to a} f\Big(g(x)\Big) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$

A few examples of the theorem above are provided below.

► **Example 1.4.13.** Evaluate $\lim_{x\to\infty} e^{\frac{1}{x}}$

[3] discussed later in this chapter

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Solution. Here,

$$f(x) = e^x \qquad \text{and} \qquad g(x) = \frac{1}{x}$$

First, find the limit of the inner function $g(x) = \dfrac{1}{x}$ as $x \to \infty$, as shown below.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Now, apply the outer function $f(x) = e^x$ to the limit of the inner function, that is, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, as shown below.

$$e^0 = 1$$

Therefore, $\lim\limits_{x\to\infty} e^{\frac{1}{x}} = e^{\lim\limits_{x\to\infty} \frac{1}{x}} = 1.$ ∎

► **Example 1.4.14.** Evaluate $\lim\limits_{x\to 1} \sqrt{x^2 + \pi}$

Solution. Here,

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = x^2 + \pi$$

First, find the limit of the inner function $g(x) = x^2 + \pi$ as $x \to 1$,

$$\lim_{x\to 1} x^2 + \pi = 1 + \pi$$

Now, apply the outer function $f(x) = \sqrt{x}$ to the limit of the inner function. That is,

$$\sqrt{1 + \pi} \approx 2.035$$

Therefore, $\lim\limits_{x\to 1} \sqrt{x^2 + \pi} = \sqrt{\lim\limits_{x\to 1} (x^2 + \pi)} \approx 2.035.$ ∎

► **Example 1.4.15.** Evaluate $\lim\limits_{x\to 1} \ln(x^2)$

Solution. Here,

$$f(x) = \ln(x) \qquad \text{and} \qquad g(x) = x^2$$

First, find the limit of the inner function $g(x) = x^2$ as $x \to 1$,

$$\lim_{x\to 1} x^2 = 1$$

Now, apply the outer function $f(x) = \ln(x)$ to the limit of the inner fnction. That is,

$$\ln(1) = 0$$

Therefore, $\lim\limits_{x\to 1} \ln(x^2) = \ln(\lim\limits_{x\to 1} x^2) = 0.$ ∎

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Chap. 1 / Sec. 1.4 / Subsec. 1.4.3 : Algebraic Manipulation

So far, we've evaluated the limit of a function by direct substitution using the limit laws. However, situations occur where applying the limit laws directly does not evaluate the limit. For example, consider the following function,

$$f(x) = \frac{\sqrt{x} - 1}{x - 1}$$

What is the limit of the function $f(x)$ as $x \to 1$? That is, $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} =?$ By direct substitution, we get,

$$\begin{aligned} \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} &= \frac{\sqrt{1} - 1}{1 - 1} \\ &= \frac{1 - 1}{1 - 1} \\ &= \frac{0}{0} \end{aligned}$$

This is absurd. In this case, direct substitution fails since the function $f(x)$ becomes undefined at $x = 1$. Although we can deal with the limit by numerical or graphical methods, a more accurate and efficient way to do that is to find a different function $g(x)$ such that it equals $f(x)$ for all values of x except at the point $x = 1$ in a way that direct substitution is applicable for $g(x)$. For example, for the function $f(x) = \frac{\sqrt{x} - 1}{x - 1}$ a function $g(x) = \frac{1}{\sqrt{x} + 1}$ can be found such that $\frac{\sqrt{x} - 1}{x - 1} = \frac{1}{\sqrt{x} + 1}$ for all $x \neq 1$. Their equivalence can be seen in the figure below,

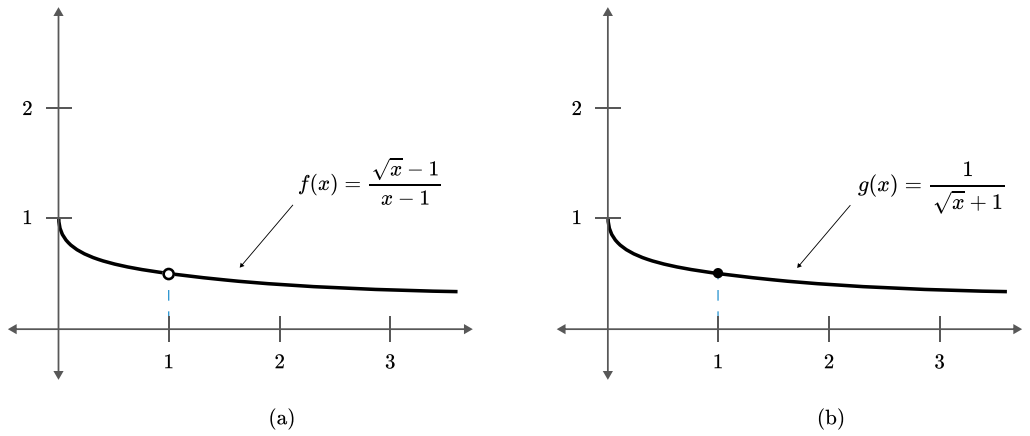

Figure 1.7

The only difference between these two functions is that $f(x)$ is not defined at the point $x = 1$, whereas $g(x)$ is defined at that point. Thus, $f(x) = g(x)$ for

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all points except $x = 1$. Because the concept of a limit focuses solely on the neighboring points of a given point, ignoring the specific behavior at that point itself, the limits of both functions are equal, as the functions happen to be equal everywhere except at $x = 1$. This allows us to write,

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1}\frac{1}{\sqrt{x}+1}\\ &= \frac{1}{\sqrt{1}+1}\\ &= \frac{1}{1+1}\\ &= \frac{1}{2}\end{aligned}$$

Therefore, $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \frac{1}{2}$. The following theorem presents the generalization of this method to evaluate the limit of a function.

Theorem 1.4.7. Let a be a real number and $f(x) = g(x)$ be functions defined on an open interval containing a, except at the point $x = a$. If

$$\lim_{x\to a} g\left(x\right) = g\left(a\right)$$

exists, then

$$\lim_{x\to a} f\left(x\right) = \lim_{x\to a} g\left(x\right) = g\left(a\right)$$

Note that in the previous example, we found the function $g(x) = \frac{1}{\sqrt{x}+1}$ as a function that equals $f\left(x\right) = \frac{\sqrt{x}-1}{x-1}$ for all x except $x = 1$. Yet, the method by which we accomplished this remains undiscussed. Relying on guesswork alone is insufficient. In fact, the expression $\frac{1}{\sqrt{x}+1}$ was actually derived from the expression $\frac{\sqrt{x}-1}{x-1}$ algebraically. Thus, it becomes necessary to be aware of certain algebraic techniques associated with finding the limit of a function analytically. We will talk about two such important techniques here. One is referred commonly as the "dividing out technique" and the other one is as the "rationalizing technique."

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· · · / Subsec. 1.4.3 / Segment 1.4.3.1 : Dividing Out Technique

The Dividing Out Technique is an essential algebraic technique used to evaluate limits when direct substitution results in an indeterminate form, such as $\frac{0}{0}$. The idea behind this technique is to simplify the expression by factoring the numerator and the denominator and then canceling out the common factors. After this simplification, we can substitute the limiting value again, often resolving the indeterminate form. For example, consider the same function as discussed in the prior section,

$$f(x) = \frac{x^2 - 4^2}{x - 4}$$

In Section 1.1, we have confirmedly concluded, through numerical and graphical methods, that the limit of $f(x) = \frac{x^2 - 4^2}{x - 4}$, as $x \to 4$, is 8. However, we now evaluate this limit by the analytical method as follows.

For that, first notice that the numerator of the expression $\frac{x^2 - 4^2}{x - 4}$ can be factored into $(x + 4) \cdot (x - 4)$ resulting $\frac{(x + 4) \cdot (x - 4)}{x - 4}$. Now, notice that the numerator and the denominator have a common factor of $(x - 4)$, thus they get canceled out and we're left with a new function $g(x) = x + 4$, that is equivalent to $f(x) = \frac{x^2 - 4^2}{x - 4}$ for all x but $x = 4$. Thus, by the theorem above, we write,

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 4^2}{x - 4} &= \lim_{x \to 4} (x + 4) \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

Which matches the previous answer that the limit of $f(x) = \frac{x^2 - 4^2}{x - 4}$, as $x \to 4$, is 8.

The dividing out technique thus can be generalized as follows.

Remark 1.4.1.

1. Factor numerator and denominator.
2. Cancel out any common factors.

A few examples of the remark above are provided below.

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► **Example 1.4.16.** Evaluate $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$

Solution. Substituting $x = 3$ directly into the function yields an indeterminate form,

$$\frac{3^2-9}{3-3} = \frac{0}{0}$$

Now, to solve the problem notice that the numerator x^2-9 can be factored into $(x+3)(x-3)$. Thus,

$$\begin{aligned}\frac{x^2-9}{x-3} &= \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} \\ &= x+3 \qquad\qquad \text{for } x \neq 3\end{aligned}$$

Now, substituting $x = 3$ into $x + 3$ yields $3 + 3 = 6$. Therefore,

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} (x+3) = 6$$

∎

► **Example 1.4.17.** Evaluate $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$

Solution. Substitution $x = 1$ directly into the function yields an indeterminate form of $\dfrac{0}{0}$. So, to solve the problem notice that the numerator $x^3 - 1$ can be factored into $(x-1)(x^2+x+1)$. Thus,

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} \\ &= \lim_{x\to 1} (x^2+x+1) \\ &= 3\end{aligned}$$

Therefore, $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = 3$ ∎

► **Example 1.4.18.** Evaluate $\lim_{x\to 1} \dfrac{x^3-6x^2+11x-6}{x-1}$

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Solution. Substitution $x = 1$ directly into the function yields division by zero, which is not defined. So, to solve the problem notice that the numerator $x^3 - 6x^2 + 11x - 6$ can be factored into $(x-1)(x-2)(x-3)$. Thus,

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-6x^2+11x-6}{x-1} &= \lim_{x\to 1}\frac{\cancel{(x-1)}(x-2)(x-3)}{\cancel{(x-1)}}\\ &= \lim_{x\to 1}(x-2)(x-3)\\ &= 2\end{aligned}$$

Therefore, $\lim_{x\to 1}\dfrac{x^3-6x^2+11x-6}{x-1} = 2$ ∎

⋯ / Subsec. 1.4.3 / Segment 1.4.3.2 : Rationalizing Technique

The rationalizing technique is a technique used for simplifying limits that involve radicals and is used when direct substitution results in an indeterminate form, such as $\frac{0}{0}$. By multiplying the expression by the conjugate of the radical, we can eliminate the radical, simplify the expression, and solve the limit. For example, consider the previous function,

$$f(x) = \frac{\sqrt{x}-1}{x-1}$$

Previously, we've seen that $\dfrac{\sqrt{x}-1}{x-1} \to \dfrac{1}{2}$, as $x \to 1$. It was determined by an equivalent function $g(x) = \dfrac{1}{\sqrt{x}+1}$. It is now shown how the expression $\dfrac{1}{\sqrt{x}+1}$ is derived from the expression $\dfrac{\sqrt{x}-1}{x-1}$.

To do that, notice that the numerator of the expression $\dfrac{\sqrt{x}-1}{x-1}$ contains a radical. In order to remove it, multiply the numerator and denominator by the conjugate of $\sqrt{x}-1$, that is,

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1}\cdot 1 &= \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{(\sqrt{x}+1)\cdot(\sqrt{x}-1)}{(x-1)\cdot(\sqrt{x}+1)} \qquad (1)\end{aligned}$$

Notice that the numerator of equation (1) can be simplified into $(\sqrt{x})^2 - 1^2$, by the identity $(a+b)\cdot(a-b) = a^2 - b^2$. Which further simplifies to $x-1$.

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Therefore we get,

$$\frac{\sqrt{x}-1}{x-1}=\frac{(x-1)}{(x-1)\left(\sqrt{x}+1\right)}$$

Notice that we now have a common factor in the numerator and denominator which can be canceled out and be left with an expression $\frac{1}{\sqrt{x}+1}$, that is defined as $g(x)=\frac{1}{\sqrt{x}+1}$. This is precisely the function $g(x)$ that we'd previously used to express its equivalence to $f(x)=\frac{\sqrt{x}-1}{x-1}$ for all x but $x=1$.

Now taking the limit as $x\to 1$, yields the same result as before. That is,

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}&=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}\\&=\frac{1}{\sqrt{1}+1}\\&=\frac{1}{1+1}\\&=\frac{1}{2}\end{aligned}$$

The rationalizing technique now can be generalized as follows.

Remark 1.4.2.

1. Multiply both the numerator and denominator by the conjugate of the numerator.
2. Cancel out any common factors.

A few examples are provided below.

► **Example 1.4.19.** Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution. Substituting $x=0$ directly into the function yields an indeterminate form,

$$\frac{\sqrt{1+0}-\sqrt{1-0}}{0}=\frac{0}{0}$$

Now, to solve the problem we multiply the numerator and denominator by the conjugate of $\sqrt{1+x}-\sqrt{1-x}$, and simplify as follows.

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

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$$
\begin{aligned}
&= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \lim_{x\to 0} \frac{2}{\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \frac{2}{\left(\sqrt{1+0}+\sqrt{1-0}\right)} \\
&= 1
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} = 1$ ∎

► **Example 1.4.20.** Evaluate $\lim_{x\to 0} \dfrac{\sqrt{3+7x}-\sqrt{3+5x}}{x}$

Solution. Substituting $x = 0$ directly into the function yields an indeterminate form,

$$\frac{\sqrt{3+7\cdot 0}-\sqrt{3+5\cdot 0}}{0} = \frac{0}{0}$$

Now, to solve the problem we multiply the numerator and denominator by the conjugate of $\sqrt{3+7x}-\sqrt{3+5x}$, and simplify as follows.

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3+7x}-\sqrt{3+5x}}{x} &= \lim_{x\to 0} \frac{\left(\sqrt{3+7x}-\sqrt{3+5x}\right)\left(\sqrt{3+7x}+\sqrt{3+5x}\right)}{x\left(\sqrt{3+7x}+\sqrt{3+5x}\right)} \\
&= \lim_{x\to 0} \frac{(3+7x)-(3+5x)}{x\left(\sqrt{3+7x}+\sqrt{3+5x}\right)} \\
&= \lim_{x\to 0} \frac{7x-5x}{x\left(\sqrt{3+7x}+\sqrt{3+5x}\right)} \\
&= \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt{3+7x}+\sqrt{3+5x}\right)} \\
&= \lim_{x\to 0} \frac{2}{\sqrt{3+7x}+\sqrt{3+5x}} \\
&= \frac{2}{\sqrt{3+7\cdot 0}+\sqrt{3+5\cdot 0}} \\
&= \frac{2}{2\sqrt{3}} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \dfrac{\sqrt{3+7x}-\sqrt{3+5x}}{x} = \dfrac{1}{\sqrt{3}}$ ∎

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► **Example 1.4.21.** Evaluate $\lim_{x\to 0} \dfrac{x}{\sqrt{1+x}-1}$

Solution. Substituting $x = 0$ directly into the function yields an indeterminate form,

$$\frac{0}{\sqrt{1+0}-1} = \frac{0}{0}$$

Now, to solve the problem we multiply the numerator and denominator by the conjugate of $\sqrt{1+x}-1$, which is $\sqrt{1+x}+1$, and simplify as follows.

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x\to 0} \frac{x\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)} \\
&= \lim_{x\to 0} \frac{x\left(\sqrt{1+x}+1\right)}{1+x-1} \\
&= \lim_{x\to 0} \frac{\cancel{x}\left(\sqrt{1+x}+1\right)}{\cancel{x}} \\
&= \lim_{x\to 0} \left(\sqrt{1+x}+1\right) \\
&= \sqrt{1+0}+1 \\
&= 2
\end{aligned}$$

Therefore, $\lim_{x\to 0} \dfrac{x}{\sqrt{1+x}-1} = 2$ ∎

► **Example 1.4.22.** Evaluate $\lim_{x\to 0} \dfrac{\sqrt{a+bx}-\sqrt{a+dx}}{x}$

Solution. Substituting $x = 0$ directly into the function yields an indeterminate form,

$$\frac{\sqrt{a+b\cdot 0}-\sqrt{a+d\cdot 0}}{0} = \frac{0}{0}$$

Now, to solve the problem we multiply the numerator and denominator by the conjugate of $\sqrt{a+bx}-\sqrt{a+dx}$, which is $\sqrt{a+bx}+\sqrt{a+dx}$, and simplify as follows.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{a+dx}}{x} &= \lim_{x\to 0} \frac{\left(\sqrt{a+bx}-\sqrt{a+dx}\right)\left(\sqrt{a+bx}+\sqrt{a+dx}\right)}{x\left(\sqrt{a+bx}+\sqrt{a+dx}\right)} \\
&= \lim_{x\to 0} \frac{(a+bx)-(a+dx)}{x\left(\sqrt{a+bx}+\sqrt{a+dx}\right)} \\
&= \lim_{x\to 0} \frac{bx-dx}{x\left(\sqrt{a+bx}+\sqrt{a+dx}\right)}
\end{aligned}$$

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$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\cancel{x}(b-d)}{\cancel{x}\left(\sqrt{a+bx}+\sqrt{a+dx}\right)} \\
&= \lim_{x\to 0} \frac{b-d}{\sqrt{a+bx}+\sqrt{a+dx}} \\
&= \frac{b-d}{\sqrt{a+0}+\sqrt{a+0}} \\
&= \frac{b-d}{2\sqrt{a}}
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \dfrac{\sqrt{a+bx}-\sqrt{a+dx}}{x} = \dfrac{b-d}{2\sqrt{a}}$ ∎

Remark 1.4.3.

$$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{a+dx}}{x} = \frac{b-d}{2\sqrt{a}}$$
$$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{a-dx}}{x} = \frac{b+d}{2\sqrt{a}}$$

Chap. 1 / Sec. 1.4 / Subsec. 1.4.4 : The Squeeze/Sandwich/Pinching Theorem

Up to this point, we have discussed techniques applicable to various functions for evaluating the limit of a function. Yet, these approaches alone often fall short in evaluating limits analytically. Thus, in this part of the section, we'll be introduced to a new technique often referred to as the "Squeeze Theorem", synonymous the "Sandwich Theorem" or occasionally the "Pinching Theorem".

The squeeze theorem might seem convoluted, however, the idea behind is quite simple and is synonymous with the following analogy.

Imagine a race competition among three cat breeds: Bengal, Turkish Angora, and Ragdoll, each completing to showcase their speed. While they share a comparable running pace on average, however the Bengal cat stands out as particularly athletic due to its natural traits. Therefore, the premise is that the Bengal cat outpaces the Turkish Angora, which in turn outpaces the Ragdoll. Thus, the hierarchical speed order among them can be shown by the following

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inequality.

$$\text{Bengal Cat} \geq \text{Turkish Angora} \geq \text{Ragdoll} \tag{1}$$

Unfortunately, on the official attempt, the Turkish Angora couldn't make it to the race due to some unforeseen circumstances, meanwhile the Bengle cat impressed everyone with an impressive speed of 32 kilometers per hour. However, to everyone's surprise, the well-trained Ragdoll cat was able to keep pace with the Bengal cat, matching its exact speed during the official race. So, by the premise, it's reasonable to conclude that had the Turkish Angora participated, it would have also run at the same speed. The rationale behind this conclusion is due to the premise that the Turkish Angora cannot surpass the Bengal cat in speed, nor can it lag behind the Ragdoll. If both cat breeds have equal speeds, it follows that the Turkish Angora—being trapped in the middle of the inequality (1)—must also share the same speed. Therefore, it is concluded that if the Turkish Angora had been in the competition, it would also have run at 32 kilometers per hour.

Similar to the analogy above, the Squeeze Theorem is based on the idea that a function $f(x)$ is squeezed between two other functions $h(x)$ and $g(x)$ near a point $x = a$. If the functions $h(x)$ and $g(x)$ have the same limit L at the point $x = a$, then the function $f(x)$ being trapped between $h(x)$ and $g(x)$ must also have the same limit L, at the point $x = a$. This can be better understood by looking at the figure below.

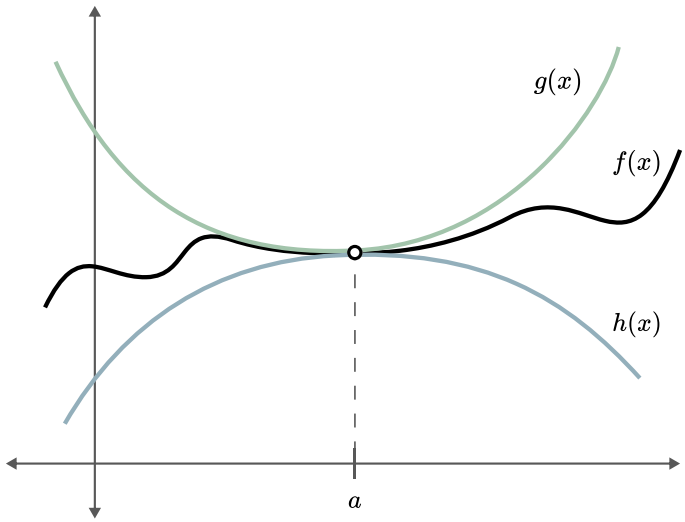

Figure 1.8

Now, this is interesting. Because, no matter how complicated $f(x)$ is, if $h(x)$ and

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$g(x)$ are relatively simple functions and their limits are equal at a point $x = a$ where $f(x)$ is squeezed between them near that point, then evaluating the limit of $f(x)$ at that point becomes trivial, as it doesn't require explicit evaluation of the limit of $f(x)$.

Thus, we have the Squeeze Theorem as follows.

Theorem 1.4.8. (The Squeeze Theorem)

If $h(x) \leq f(x) \leq g(x)$ for all x over an open interval containing a, except possibly at a itself, and if the following is true,

$$\lim_{x \to a} h\left(x\right) = \lim_{x \to a} g\left(x\right) = L$$

then,

$$\lim_{x \to a} f\left(x\right) = L$$

This theorem allows us to analytically evaluate functions that were not supposed to be evaluated by other methods discussed before. For example, consider the following function,

$$f(x) = \frac{\sin\left(x\right)}{x}$$

What is the limit of $f(x)$, as $x \to 0$?

If we proceed with the previous methods, we get,

$$\begin{aligned} \lim_{x \to 0} \frac{\sin\left(x\right)}{x} &= \frac{\sin\left(0\right)}{0} \\ &= \frac{0}{0} \end{aligned}$$

Which is absurd. However, by the squeeze theorem, we can evaluate this limit analytically. Before that, we must recall that,

$$\begin{aligned} \text{the area of triangle} &= \frac{1}{2}\left(\text{base} \times \text{height}\right) \\ \text{the area of a sector} &= \frac{1}{2} r^2 \theta \end{aligned}$$

where, r is the radius and θ is the angle.

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With that, we draw the following figure of a unit circle to evaluate the limit $\lim_{x\to 0}\frac{\sin(x)}{x}$. Note that, since it is a unit circle, therefore the radius $r = 1$.

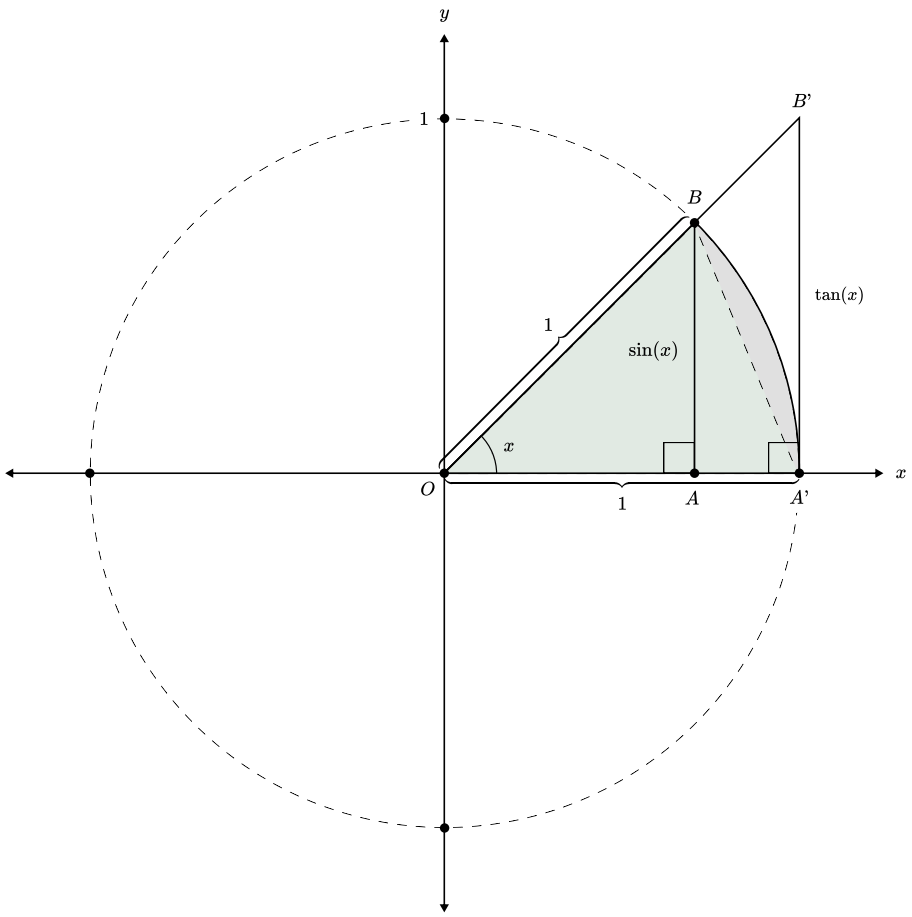

Figure 1.9

In the figure, we see that, for ΔAOB,

$$\begin{aligned}\sin(x) &= \frac{\text{opposite side}}{\text{hypotenuse}}\\ &= \frac{\text{opposite side}}{1}\\ &= \text{opposite side}\\ \therefore \text{opposite side} &= \sin(x)\end{aligned}$$

For $\Delta A'OB'$,

$$\begin{aligned}\tan(x) &= \frac{\text{opposite side}}{\text{adjacent side}}\\ &= \frac{\text{opposite side}}{1}\end{aligned}$$

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$$= \text{opposite side}$$
$$\therefore \text{opposite side} = \tan(x)$$

Now, referring to Figure 1.9, it's evident that,

$$\text{area of } \Delta OA'B \leq \text{area of sector } OA'B \leq \text{area of } \Delta OA'B' \tag{1}$$

where,

$$\begin{aligned}
\text{area of } \Delta OA'B &= \frac{1}{2}(\text{base} \times \text{height}) &&= \frac{1}{2} \cdot 1 \cdot \sin(x) &&= \frac{\sin(x)}{2} \\
\text{area of sector } OA'B &= \frac{1}{2} r^2 \cdot \theta &&= \frac{1}{2}(1)^2 \cdot x &&= \frac{x}{2} \\
\text{area of } \Delta OA'B' &= \frac{1}{2}(\text{base} \times \text{height}) &&= \frac{1}{2} \cdot 1 \cdot \tan(x) &&= \frac{\tan(x)}{2}
\end{aligned}$$

Thus, inequality (1) can be written as,

$$\frac{\sin(x)}{2} \leq \frac{x}{2} \leq \frac{\tan(x)}{2}$$

Now, dividing it by $\frac{\sin(x)}{2}$ yields,

$$\begin{aligned}
\frac{\cancel{\frac{\sin(x)}{2}}}{\cancel{\frac{\sin(x)}{2}}} &\leq \frac{\frac{x}{2}}{\frac{\sin(x)}{2}} \leq \frac{\frac{\tan(x)}{2}}{\frac{\sin(x)}{2}} \\
1 &\leq \frac{x}{\sin(x)} \leq \frac{\tan(x)}{\sin(x)} \\
1 &\leq \frac{x}{\sin(x)} \leq \frac{\frac{\sin(x)}{\cos(x)}}{\sin(x)} \\
1 &\leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}
\end{aligned}$$

Now, by taking the reciprocals, we get,

$$1 \geq \frac{\sin(x)}{x} \geq \cos(x)$$

Now, taking the limit as $x \to 0$, we get,

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$$\lim_{x\to 0} 1 \geq \lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) \geq \lim_{x\to 0}\cos(x)$$
$$1 \geq \lim_{x\to 0}\frac{\sin(x)}{x} \geq 1$$

Therefore, by the Squeeze Theorem, we get,

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

The limit above is crucial because it frequently aids in evaluating other limits of functions using the Squeeze Theorem. Therefore, we have the following remark.

Remark 1.4.4. (An Important Limit)

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

A few examples of the Squeeze Theorem are provided below.

► **Example 1.4.23.** Evaluate $\lim_{x\to 0}\dfrac{x}{\sin(x)}$

Solution. We know that $\lim_{x\to 0}\dfrac{\sin(x)}{x} = 1$, therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(x)} &= \lim_{x\to 0}\frac{1}{\frac{\sin(x)}{x}}\\ &= \frac{1}{\lim_{x\to 0}\frac{\sin(x)}{x}}\\ &= \frac{1}{1}\end{aligned}$$

Therefore, $\lim_{x\to 0}\dfrac{x}{\sin(x)} = 1$ ∎

Remark 1.4.5.

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{x}{\sin(x)} = 1$$

► **Example 1.4.24.** Evaluate $\lim_{x\to 0}\dfrac{\tan(x)}{x}$

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Solution. We know that $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\lim_{x\to 0}\cos(x)=1$, therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)}{x}&=\lim_{x\to 0}\frac{\frac{\sin(x)}{\cos(x)}}{x}\\&=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{x}\\&=\lim_{x\to 0}\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{x}\\&=\lim_{x\to 0}\frac{1}{\cos(x)}\cdot\lim_{x\to 0}\frac{\sin(x)}{x}\\&=1\end{aligned}$$

Therefore, $\lim_{x\to 0}\frac{\tan(x)}{x}=1$ ∎

► **Example 1.4.25.** Evaluate $\lim_{x\to 0}\frac{x}{\tan(x)}$

Solution. We know that $\lim_{x\to 0}\frac{\tan(x)}{x}=1$, therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\tan(x)}&=\lim_{x\to 0}\frac{1}{\frac{\tan(x)}{x}}\\&=\frac{1}{\lim_{x\to 0}\frac{\tan(x)}{x}}\\&=1\end{aligned}$$

Therefore, $\lim_{x\to 0}\frac{x}{\tan(x)}=1$ ∎

Remark 1.4.6.

$$\lim_{x\to 0}\frac{\tan(x)}{x}=\lim_{x\to 0}\frac{x}{\tan(x)}=1$$

► **Example 1.4.26.** Evaluate $\lim_{x\to 0}\frac{\cos(x)-1}{x}$

Solution. Since $\lim_{x\to 0}\frac{\sin(x)}{x}=1$, therefore,

$$\lim_{x\to 0}\frac{\cos(x)-1}{x}=\lim_{x\to 0}\left(\frac{\cos(x)-1}{x}\cdot\frac{\cos(x)+1}{\cos(x)+1}\right)$$

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$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\cos^2(x)-1}{x\cdot(\cos(x)+1)} \\
&= \lim_{x\to 0} \frac{-\sin^2(x)}{x\cdot(\cos(x)+1)} \\
&= -\lim_{x\to 0} \left(\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{\cos(x)+1}\right) \\
&= -1\cdot\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \dfrac{\cos(x)-1}{x}$. ∎

Remark 1.4.7.

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

► **Example 1.4.27.** Evaluate $\lim_{x\to\infty} \dfrac{\sin(x)}{x}$

Solution. We know that for all x, the sine function oscillates between -1 to 1. Therefore,

$$
\begin{aligned}
-1 \le \sin(x) &\le 1 \\
-\frac{1}{x} \le \frac{\sin(x)}{x} &\le \frac{1}{x} \\
-\lim_{x\to\infty}\frac{1}{x} \le \lim_{x\to\infty}\frac{\sin(x)}{x} &\le \lim_{x\to\infty}\frac{1}{x} \\
0 \le \lim_{x\to\infty}\frac{\sin(x)}{x} &\le 0
\end{aligned}
$$

As a result, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$$

∎

► **Example 1.4.28.** Evaluate $\lim_{x\to\infty} \dfrac{\cos(x)}{x}$

Solution. We know that for all x, the sine function oscillates between -1 to 1. Therefore,

$$-1 \le \cos(x) \le 1$$

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$$-\frac{1}{x} \leq \frac{\cos(x)}{x} \leq \frac{1}{x}$$
$$-\lim_{x\to\infty} \frac{1}{x} \leq \lim_{x\to\infty} \frac{\cos(x)}{x} \leq \lim_{x\to\infty} \frac{1}{x}$$
$$0 \leq \lim_{x\to\infty} \frac{\cos(x)}{x} \leq 0$$

As a result, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\cos(x)}{x} = 0$$

∎

Remark 1.4.8.

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = \lim_{x\to\infty} \frac{\cos(x)}{x} = 0$$

Chap. 1 / Sec. 1.4 / Subsec. 1.4.5 : Trigonometric Manipulation

In addition to the technique discussed above, trigonometric manipulation is a crucial technique in solving limit problems involving trigonometric functions. It often requires the use of specific trigonometric identities, algebraic simplifications, and limit theorems to simplify expressions that would otherwise be challenging to evaluate directly. The goal is to transform the original expression into a form where the limit can be easily computed, often by eliminating indeterminate forms such as $\frac{0}{0}$. The following trigonometric identities are commonly used.

1. Reciprocal Identities

$$\sin(x) = \frac{1}{\csc(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$
$$\cos(x) = \frac{1}{\sec(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\tan(x) = \frac{1}{\cot(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

2. Tangent and Cotangent Identities

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$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

3. Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1 \qquad 1 + \cot^2(x) = \csc^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

4. Reduction Formulas

$$\sin(-x) = -\sin(x) \qquad \csc(-x) = -\csc(x)$$

$$\cos(-x) = \cos(x) \qquad \sec(-x) = \sec(x)$$

$$\tan(-x) = -\tan(x) \qquad \cot(-x) = -\cot(x)$$

5. Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x) \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc(x)$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x) \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan(x)$$

6. Sum and Difference Formulas

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$

7. Double Angle Formulas

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

8. Power Reducing Formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

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$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$$

9. Sum to Product Formulas

$$\sin(x)+\sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x)-\sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x)+\cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x)-\cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

10. Product to Sum Formulas

$$\sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$$

$$\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos(x)\sin(y) = \frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

A few examples using the identities and formulas above are provided below.

► **Example 1.4.29.** Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin(x)}{\cos(x)}$

Solution. In order to evaluate the limit, we first simplify it as follows.

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)}{\cos(x)} &= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)}{\cos(x)}\cdot\frac{1+\sin(x)}{1+\sin(x)}\\ &= \lim_{x\to\frac{\pi}{2}} \frac{(1+\sin(x))(1-\sin(x))}{(1+\sin(x))\cos(x)}\\ &= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin^2(x)}{(1+\sin(x))\cos(x)}\end{aligned}$$

Since, $\sin^2(x)+\cos^2(x)=1$, therefore, we write,

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin^2(x)}{(1+\sin(x))\cos(x)} = \lim_{x\to\frac{\pi}{2}} \frac{\cos^2(x)}{(1+\sin(x))\cos(x)}$$

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$$
\begin{aligned}
&= \lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{1+\sin(x)} \\
&= \frac{\cos\left(\frac{\pi}{2}\right)}{1+\sin\left(\frac{\pi}{2}\right)} \\
&= \frac{0}{1+1} \\
&= 0
\end{aligned}
$$

Therefore, $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)}{\cos(x)} = 0$ ∎

► **Example 1.4.30.** Evaluate $\lim_{x\to 0} (\csc(x) - \cot(x))$

Solution. In order to evaluate the limit, we simplify it as follows,

$$
\begin{aligned}
\lim_{x\to 0} (\csc(x) - \cot(x)) &= \lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right) \\
&= \lim_{x\to 0} \frac{1-\cos(x)}{\sin(x)}
\end{aligned}
$$

Now, it is known, from the double angle identity, that $\cos(2x) = 1 - 2\sin^2(x)$, therefore,

$$
\begin{aligned}
\cos(x) &= 1 - 2\sin^2\left(\frac{x}{2}\right) \\
\cos(x) - 1 &= -2\sin^2\left(\frac{x}{2}\right) \\
1 - \cos(x) &= 2\sin^2\left(\frac{x}{2}\right)
\end{aligned}
$$

Similarly, it is know, from the double angle identity, that $\sin(2x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$, therefore,

$$
\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)
$$

This implies that,

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{\sin(x)} &= \lim_{x\to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \\
&= \lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}
\end{aligned}
$$

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$$\begin{aligned}
&= \lim_{x\to 0} \tan\left(\frac{x}{2}\right)\\
&= \tan\left(\frac{0}{2}\right)\\
&= 0
\end{aligned}$$

Therefore, $\lim\limits_{x\to 0} (\csc(x) - \cot(x)) = 0$ ∎

► **Example 1.4.31.** Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{\tan(x)}\right)$

Solution. In order to evaluate the limit, we first simplify it as follows.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{\tan(x)}\right) &= \lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right)\\
&= \lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)}
\end{aligned}$$

Now, since $\cos(2x) = 1 - 2\sin^2(x)$, therefore, $\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$. Likewise, since $\sin(2x) = 2\sin(x)\cos(x)$, therefore, $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$. Thus,

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)} &= \lim_{x\to 0}\frac{1-\left(1-2\sin^2\left(\frac{x}{2}\right)\right)}{2\sin\left(\frac{x}{x}\right)\cos\left(\frac{x}{2}\right)}\\
&= \lim_{x\to 0}\frac{\cancel{2}\sin^2\left(\frac{x}{2}\right)}{\cancel{2}\sin\left(\frac{x}{x}\right)\cos\left(\frac{x}{2}\right)}\\
&= \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\\
&= \lim_{x\to 0}\tan\left(\frac{x}{2}\right)\\
&= \tan\left(\frac{0}{2}\right)\\
&= 0
\end{aligned}$$

Therefore, $\lim\limits_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{\tan(x)}\right) = 0$ ∎

Some limit problems necessitate a combination of algebraic and trigonometric manipulations to solve them. This is shown in the following examples.

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► **Example 1.4.32.** Evaluate $\lim_{x\to 0}\dfrac{1-\cos(x)}{x}$

Solution. Since it is known, from the double angle identity, that $\cos(2x) = 1 - 2sin^2(x)$, therefore,

$$\begin{aligned}\cos(x) &= 1 - 2\sin^2\left(\frac{x}{2}\right)\\ \cos(x) - 1 &= -2\sin^2\left(\frac{x}{2}\right)\\ 1 - \cos(x) &= 2\sin^2\left(\frac{x}{2}\right)\end{aligned}$$

This implies that,

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x}$$

Now, we do the following simplification to apply the standard limit $\lim_{x\to 0}\dfrac{\sin(x)}{x} = 1$. That is,

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{x} &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x}\\ &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x}\cdot\frac{x}{x}\\ &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}\cdot x\\ &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}\cdot 4}\cdot x\\ &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)^2\cdot 4}\cdot x\\ &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)^2}\cdot\frac{x}{4}\\ &= \lim_{x\to 0}\cancel{2}\cdot\left[\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)^2}\right]\cdot\frac{x}{\cancel{4}}\end{aligned}$$

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$$= \lim_{x\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)}\right]^2 \cdot \frac{x}{2}$$

$$= \lim_{x\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)}\right]^2 \cdot \lim_{x\to 0}\frac{x}{2}$$

Now, as $x \to 0$, $\frac{x}{2} \to 0$, since x is multiplied by a constant of $\frac{1}{2}$. Therefore, by the the standard limit $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, we get the following,

$$\lim_{x\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)}\right]^2 \cdot \lim_{x\to 0}\frac{x}{2} = \underbrace{\lim_{\frac{x}{2}\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)}\right]^2}_{1} \cdot \underbrace{\lim_{\frac{x}{2}\to 0}\frac{x}{2}}_{0}$$

$$= 1 \cdot 0$$

$$= 0$$

Therefore, $\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$ ∎

► **Example 1.4.33.** Evaluate $\lim_{x\to 0}\frac{\cos(2x)-\cos(3x)}{x^2}$

Solution. Using the Sum to Product Formula $\cos(x)-\cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$, we write the following,

$$\lim_{x\to 0}\frac{\cos(2x)-\cos(3x)}{x^2} = \lim_{x\to 0}\frac{-2\sin\left(\frac{2x+3x}{2}\right)\sin\left(\frac{2x-3x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{5x}{2}\right)\sin\left(\frac{x}{2}\right)}{x^2}$$

Now, we do the following simplification to apply the standard limit $\lim_{x\to 0}\frac{\sin(x)}{x} =$

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1.

$$\begin{aligned}
\lim_{x\to 0}\frac{2\sin\left(\frac{5x}{2}\right)\sin\left(\frac{x}{2}\right)}{x^2} &= \lim_{x\to 0}\frac{2\left[\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\frac{5x}{2}\right]\cdot\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{x}{2}\right]}{x^2}\\
&= \lim_{x\to 0}\frac{2\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5x}{2}\cdot\frac{x}{2}}{x^2}\\
&= \lim_{x\to 0}\left(\cancel{2}\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5x^2}{\cancel{4}}\right)\cdot\frac{1}{x^2}\\
&= \lim_{x\to 0}\left(\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5\cancel{x^2}}{2}\cdot\frac{1}{\cancel{x^2}}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5}{2}\right)\\
&= \lim_{x\to 0}\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5}{2}
\end{aligned}$$

Now, as $x\to 0$, both $\frac{5x}{2}\to 0$ and $\frac{x}{2}\to 0$, since they are multiplied by a constant of $\frac{5}{2}$ and $\frac{1}{2}$, respectively. Therefore, by the the standard limit $\lim_{x\to 0}\frac{\sin(x)}{x}=1$, we write,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5}{2} &= \lim_{\frac{5x}{2}\to 0}\frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}}\cdot\lim_{\frac{x}{2}\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\cdot\frac{5}{2}\\
&= 1\cdot 1\cdot\frac{5}{2}\\
&= \frac{5}{2}
\end{aligned}$$

Therefore, $\lim_{x\to 0}\frac{\cos(2x)-\cos(3x)}{x^2}=\frac{5}{2}$ ∎

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———————The End of Section 1.4
Analytic Evaluation of Limit of a Function

§ 1.5 Limits Involving ∞

Limits of functions can involve infinity and they can do so in two common ways. First, the function $f(x)$ can increase or decrease without a bound as its input x approaches a finite value, these limits are referred to as "Infinite Limits." Second, the input x can increase or decrease without bound, these are referred to as "Limits at Infinity." The concepts behind these two are rather distinct. For example, consider the function we'd seen before.

$$f(x) = \frac{1}{(x-4)^2}$$

The graph of $f(x)$ is shown below.

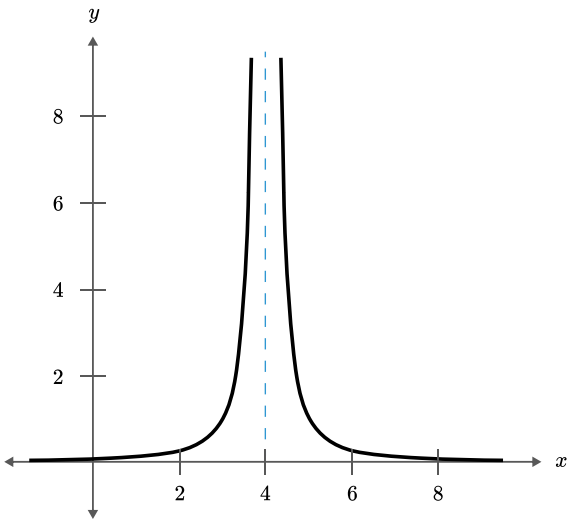

Figure 1.10

Based on that, consider the following limits,

$$1. \quad \lim_{x\to 4} \frac{1}{(x-4)^2} = \infty \qquad\qquad 2. \quad \lim_{x\to\infty} \frac{1}{(x-4)^2} = 0$$

In the first limit, when the input x approaches a finite value of 4 from both sides, the function $f(x)$ increases without a bound and thus the limit does not exist.

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On the other hand, in the second limit, as x increases without a bound, $f(x)$ approaches a finite value of 0, and thus the limit, in this case, does exist.

The first limit is an example of an "infinite limit," whereas the second one is of "limit at infinity."

In general, with an infinite limit, the dependent variable approaches infinity as the independent variable approaches a finite value. With a limit at infinity, the dependent variable approaches a finite value as the independent variable approaches infinity.

Chap. 1 / Sec. 1.5 / Subsec. 1.5.1 : Infinite Limits

Given our understanding of infinite limits, we have the following definition.

Definition 1.5.1. (Infinite Limits) Let $f(x)$ be a function defined for all x in an open interval containing a. Then,

1. If $f(x)$ increases without a bound as $x \to a$, where, $x \neq a$, then we write,
$$\lim_{x \to a} f(x) = +\infty$$
2. If $f(x)$ decreases without a bound as $x \to a$, where, $x \neq a$, then we write,
$$\lim_{x \to a} f(x) = -\infty$$

In both of these cases, *the limit does not exist.*

Note that, since the symbol ∞ does not represents a number, therefore the limits $\lim_{x \to a} f(x) = +\infty$ and $\lim_{x \to a} f(x) = -\infty$ does not imply that the limit exists, but rather denotes the behavior of the limit (whether the function is growing in a positive or negative direction). Definition above can be elaborated in detail through the use of one-sided limits as follows.

Definition 1.5.2. (One-sided Infinite Limits)

1. Infinite limits from the left: Let $f(x)$ be a function defined for all x near a. Then,

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(a) If $f(x)$ increases without a bound as $x \to a$, where, $x < a$, then

$$\lim_{x \to a^-} f(x) = +\infty$$

(b) If $f(x)$ decreases without a bound as $x \to a$, where, $x < a$, then

$$\lim_{x \to a^-} f(x) = -\infty$$

2. Infinite limits from the right: Let $f(x)$ be a function defined for all x near a. Then,

(a) If $f(x)$ increases without a bound as $x \to a$, where, $x > a$, then

$$\lim_{x \to a^+} f(x) = +\infty$$

(b) If $f(x)$ decreases without a bound as $x \to a$, where, $x > a$, then

$$\lim_{x \to a^+} f(x) = -\infty$$

Consider the following figures to better understand the definition above.

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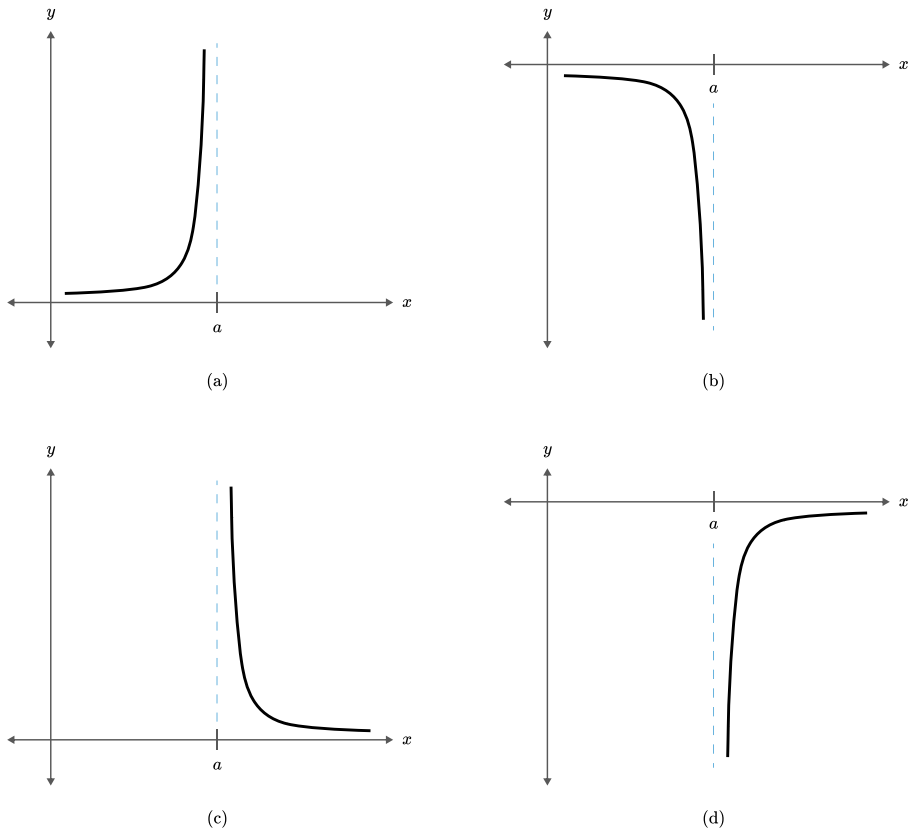

Figure 1.11

Figure 1.11(a) shows the limit $\lim_{x\to a^-} f(x) = +\infty$, Figure 1.11(b) shows the limit $\lim_{x\to a^-} f(x) = -\infty$, Figure 1.11(c) shows the limit $\lim_{x\to a^+} f(x) = +\infty$, lastly, Figure 1.11(d) shows the limit $\lim_{x\to a^+} f(x) = -\infty$.

Notice that, as $x \to a$, the function $f(x)$ approaches a vertical line at $x = a$ in all of the infinite limits shown in Figure 1.11. The vertical line at the point $x = a$ is called a *vertical asymptote*. It is a line that is approached by the function $f(x)$ when $x \to a$.

Definition 1.5.3. (Vertical Asymptote)
If $f(x)$ approaches positive infinity or negative infinity as $x \to a$ from the right or the left side, then the line $x = a$ is called a vertical asymptote of $f(x)$.

We now began to evaluate functions that have an infinite limit. However, before that, notice that, if we have a fraction, $\frac{a}{b}$, where $a \neq 0$ and $b \neq 0$, then

1. if $b \to 0$, then $\frac{a}{b} \to \infty$.

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2. if $b \to \infty$, then $\dfrac{a}{b} \to 0$.

► **Example 1.5.1.** Evaluate $\lim\limits_{x \to 0^+} \dfrac{x^2+1}{x}$

Solution. Notice that as $x \to 0^+$, the denominator of the function $\dfrac{x^2+1}{x}$ approaches 0, whereas the numerator (x^2+1) approaches the value 1. Therefore,

$$\lim_{x \to 0^+} \frac{x^2+1}{x} = \infty$$

Therefore, $\lim\limits_{x \to 0^+} \dfrac{x^2+1}{x} = \infty$ ∎

► **Example 1.5.2.** Evaluate $\lim\limits_{x \to 0^+} \cot(x)$.

Solution. We know that,

$$\lim_{x \to 0^+} \cot(x) = \lim_{x \to 0^+} \frac{\cos(x)}{\sin(x)}$$

We know further that as $x \to 0^+$, $\cos(x) \to 1$ and $\sin(x) \to 0$, where, $\sin(x) > 0$. Therefore,

$$\begin{aligned} \lim_{x \to 0^+} \cot(x) &= \lim_{x \to 0^+} \frac{\cos(x)}{\sin(x)} \\ &= +\infty \end{aligned}$$

Therefore, $\lim\limits_{x \to 0^+} \cot(x) = +\infty$ ∎

► **Example 1.5.3.** Evaluate $\lim\limits_{x \to 0^-} \cot(x)$.

Solution. We know that,

$$\lim_{x \to 0^-} \cot(x) = \lim_{x \to 0^-} \frac{\cos(x)}{\sin(x)}$$

We know further that as $x \to 0^-$, and $\sin(x) \to 0$ and $\cos(x) \to 1$, where, $\sin(x) < 0$. Therefore,

$$\begin{aligned} \lim_{x \to 0^-} \cot(x) &= \lim_{x \to 0^-} \frac{\cos(x)}{\sin(x)} \\ &= -\infty \end{aligned}$$

Therefore, $\lim\limits_{x \to 0^-} \cot(x) = -\infty$ ∎

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Chap. 1 / Sec. 1.5 / Subsec. 1.5.2 : Limit at Infinity

The limit at infinity suggests that the limit of a function will have a finite value as its input approaches positive or negative infinity. For example, consider the following function,

$$f(x) = \pi + \frac{1}{x}$$

If $x \to +\infty$, what value does $f(x)$ approach? To determine, notice that as $x \to +\infty$, the term $\dfrac{1}{x} \to 0$ in the function above. Therefore,

$$\begin{aligned}\lim_{x\to+\infty}\left(\pi + \frac{1}{x}\right) &= \lim_{x\to+\infty}\pi + \lim_{x\to+\infty}\frac{1}{x}\\ &= \pi + 0\\ &= \pi\end{aligned}$$

Similarly, if $x \to -\infty$, the term $\dfrac{1}{x} \to 0$, that is, $\lim\limits_{x\to-\infty}\dfrac{1}{x} = 0$. Thus, we write the following,

$$\begin{aligned}\lim_{x\to-\infty}\left(\pi + \frac{1}{x}\right) &= \lim_{x\to-\infty}\pi + \lim_{x\to-\infty}\frac{1}{x}\\ &= \pi + 0\\ &= \pi\end{aligned}$$

The result can be seen visually in the following figure.

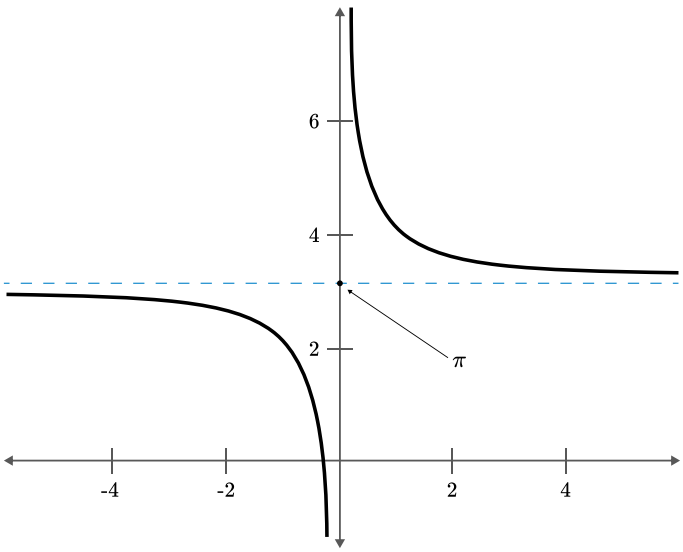

Figure 1.12

In general, we have the following definition of the limit at infinity.

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Definition 1.5.4. (Limit at Infinity) If $f(x)$ becomes arbitrarily close to L for all sufficiently large $x > 0$, then

$$\lim_{x\to+\infty} f(x) = L$$

If $f(x)$ becomes arbitrarily close to L for all sufficiently large $x < 0$, then

$$\lim_{x\to-\infty} f(x) = L$$

Notice that if $f(x) \to L$, as $x \to \pm\infty$, the graph of $f(x)$ approaches the horizontal line $y = L$. The horizontal line $y = L$ is called the horizontal asymptote of $f(x)$.

Definition 1.5.5. (Horizontal Asymptote)
If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote of $f(x)$.

For example, since $\lim_{x\to\infty}\left(\pi + \frac{1}{x}\right) = \pi$, therefore the line $y = \pi$ is the *horizontal asymptote* for the function $f(x) = \pi + \frac{1}{x}$, as it was shown in the Figure above.

► **Example 1.5.4.** Evaluate $\lim_{x\to\infty} \frac{ex+1}{x}$

Solution. In order to evaluate the limit, we do the following simplification,

$$\begin{aligned}\lim_{x\to\infty} \frac{ex+1}{x} &= \lim_{x\to\infty} \frac{ex}{x} + \frac{1}{x}\\ &= \lim_{x\to\infty} e + \frac{1}{x}\end{aligned}$$

Now, as $x \to \infty$, the term $\frac{1}{x} \to 0$. This implies that,

$$\begin{aligned}\lim_{x\to\infty} e + \frac{1}{x} &= \lim_{x\to\infty} e + \lim_{x\to\infty} \frac{1}{x}\\ &= e + 0\\ &= e\end{aligned}$$

Therefore, $\lim_{x\to\infty} \frac{ex+1}{x} = e$ ∎

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► **Example 1.5.5.** Evaluate $\lim_{x\to\infty} \dfrac{x+x^2+x^3+x^4+x^5}{x^5}$

Solution. In order to evaluate the limit, we do the following simplification,

$$\begin{aligned}\lim_{x\to\infty} \frac{x+x^2+x^3+x^4+x^5}{x^5} &= \lim_{x\to\infty} \frac{x}{x^5}+\frac{x^2}{x^5}+\frac{x^3}{x^5}+\frac{x^4}{x^5}+\frac{x^5}{x^5}\\ &= \lim_{x\to\infty} \frac{1}{x^4}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1\end{aligned}$$

Now, as $x\to\infty$, the all term except the last one approach 0 in the limit above, therefore, we write,

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x^4}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1 &= \lim_{x\to\infty} \frac{1}{x^4}+\lim_{x\to\infty} \frac{1}{x^3}+\lim_{x\to\infty} \frac{1}{x^2}+\lim_{x\to\infty} \frac{1}{x}+\lim_{x\to\infty} 1\\ &= 0+0+0+0+1\\ &= 1\end{aligned}$$

Therefore, $\lim_{x\to\infty} \dfrac{x+x^2+x^3+x^4+x^5}{x^5} = 1$ ∎

Chap. 1 / Sec. 1.5 / Subsec. 1.5.3 : Infinite Limits at Infinity and End Behaviors

If a function $f(x)$ increases or decreases without a bound as its input x increases or decreases without a bound, then we have an infinite limit at infinity. For example, consider the following function,

$$f(x) = x^3$$

In the given function, $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, that is, as $x\to+\infty$, the function $f(x)\to+\infty$. Similarly, as $x\to-\infty$, the function $f(x)\to-\infty$. This can be seen in the following figure.

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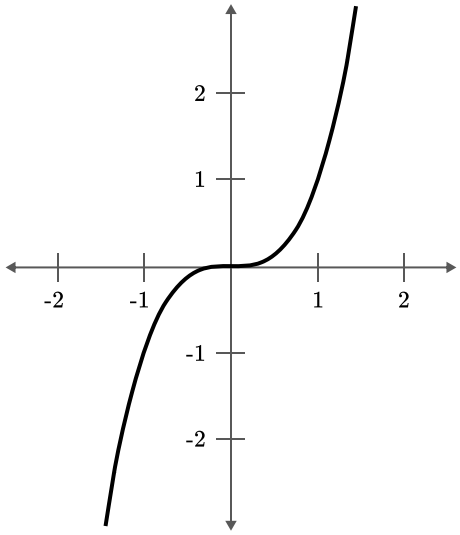

Figure 1.13

In general, we have the following definition.

Definition 1.5.6. (Infinite Limits at Infinity) If $f(x)$ increases without a bound as $x \to +\infty$ or $x \to -\infty$, then

$$\lim_{x\to+\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty$$

If $f(x)$ decreases without a bound as $x \to +\infty$ or $x \to -\infty$, then

$$\lim_{x\to+\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty$$

Having discussed the idea of the infinite limits at infinite, it is now crucial to discuss the end behavior of a function. End behavior refers to how a function $f(x)$ behaves as its input x increases or decreases indefinitely. The function can show various behaviors at its endpoints. The following are among the most common ones.

1. $f(x)$ approaches a horizontal asymptotes $y = L$.
2. $f(x) \to +\infty$ or $f(x) \to -\infty$
3. Neither 1 nor 2.

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$\cdots$ / Subsec. 1.5.3 / Segment 1.5.3.1 : The Limit of Power functions as $x \to \pm\infty$

Consider the following power function, where, n is a positive integer.

$$f(x) = x^n$$

Now, if n is even, then $f(x) \to +\infty$, as $x \to \pm\infty$. In contrast, if n is odd, then $f(x) \to +\infty$, as $x \to +\infty$ and $f(x) \to -\infty$, as $x \to -\infty$. More generally, we have the following remark.

Remark 1.5.1.

$$\lim_{x\to\infty} x^n = \infty \quad n = 1, 2, 3, \cdots \quad \text{and} \quad \lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \cdots \\ +\infty, & n = 2, 4, 6, \cdots \end{cases}$$

Now consider the reciprocal of the power function, where n is a positive integer.

$$f(x) = \frac{1}{x^n} = x^{-n}$$

For the reciprocal, as $x \to \pm\infty$, $f(x) \to 0$, regardless of whether n is even or odd.

► **Example 1.5.6.** Analyze the end behavior of $f(x) = x^3$.

Solution. Here, $n = 3$, therefore n is odd. That's why,

1. As $x \to +\infty$, $f(x) \to +\infty$
2. As $x \to -\infty$, $f(x) \to -\infty$

▮

► **Example 1.5.7.** Analyze the end behavior of $f(x) = x^4$.

Solution. Here, $n = 4$, therefore n is even. That's why,

1. As $x \to +\infty$, $f(x) \to +\infty$
2. As $x \to -\infty$, $f(x) \to +\infty$

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▮

► **Example 1.5.8.** Analyze the end behavior of $f(x) = \dfrac{1}{x^3}$.

Solution. Here, $f(x) = \dfrac{1}{x^3}$ is a reciprocal of x^3. That's why,

1. As $x \to +\infty$, $f(x) \to 0$
2. As $x \to -\infty$, $f(x) \to 0$

▮

⋯ / Subsec. 1.5.3 / Segment 1.5.3.2 : The Limit of Polynomial Functions as $x \to \pm\infty$

Now, using the power function, we can determine the behavior of any polynomial function as $x \to \pm\infty$. For that, consider the polynomial function,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where, $a_n \neq 0$ and $n \geq 1$. Now, factoring $P(x)$ yields,

$$\begin{aligned}
P(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
&= x^n \left(\frac{a_n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + \cdots + \frac{a_1 x}{x^n} + \frac{a_0}{x^n} \right) \\
&= x^n \left(a_n \frac{x^n}{x^n} + a_{n-1}\frac{x^{n-1}}{x^n} + \cdots + a_1 \frac{x}{x^n} + a_0 \frac{1}{x^n} \right) \\
&= x^n \left(a_n x^{n-n} + a_{n-1}x^{n-1-n} + \cdots + a_1 x^{1-n} + a_0 \frac{1}{x^n} \right) \\
&= x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)
\end{aligned}$$

As $x \to \pm\infty$, all the terms inside the parentheses approach 0, except the first term. Therefore,

$$\lim_{x\to\pm\infty} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \right) = \lim_{x\to\pm\infty} a_n x^n$$

This implies that as $x \to \pm\infty$, the behavior of the polynomial $P(x)$ is determined by the term with the highest power of x, that is, $a_n x^n$. Therefore, we have the following remark.

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Remark 1.5.2. The end behavior of a polynomial function $P(x)$ resembles the end behavior of its highest degree term.

A few examples are shown below.

► **Example 1.5.9.** If $P(x) = 2x^4 - 3x^3 + 5x^2 - 7x + 11$, evaluate $\lim_{x\to\pm\infty} P(x)$.

Solution. Here, $P(x) = 2x^4 - 3x^3 + 5x^2 - 7x + 11$

1. As $x \to +\infty$, the highest degree term dominates, that is,
$$\begin{aligned}\lim_{x\to+\infty} P(x) &= \lim_{x\to+\infty} 2x^4 \\ &= \infty\end{aligned}$$
2. As $x \to -\infty$, the highest degree term again dominates, that is,
$$\begin{aligned}\lim_{x\to-\infty} P(x) &= \lim_{x\to-\infty} 2x^4 \\ &= \infty\end{aligned}$$
since the degree is even.

∎

► **Example 1.5.10.** If $P(x) = -3x^5 + 5x^4 - 7x^3 + 11x^2 - 13x + 17$, evaluate $\lim_{x\to\pm\infty} P(x)$

Solution. Here, $P(x) = -3x^5 + 5x^4 - 7x^3 + 11x^2 - 13x + 17$

1. As $x \to +\infty$, the highest degree term dominates, that is,
$$\begin{aligned}\lim_{x\to+\infty} P(x) &= \lim_{x\to+\infty} \left(-3x^5\right) \\ &= -\infty\end{aligned}$$
2. As $x \to -\infty$, the highest degree term again dominates, that is,
$$\begin{aligned}\lim_{x\to-\infty} P(x) &= \lim_{x\to-\infty} \left(-3x^5\right) \\ &= \infty\end{aligned}$$
since the degree is odd and the leading coefficient is negative.

∎

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$\cdots$ / Subsec. 1.5.3 / Segment 1.5.3.3 : The Limit of Rational Functions as $x \to \pm\infty$

As of yet, we've explored the end behavior of power and polynomial functions with relative ease. However, the end behavior of rational functions isn't as straightforward as seen previously. For example, consider the following rational function,

$$f(x) = \frac{x^2 - 1}{x^3 - 1}$$

Here, as $x \to \infty$, both the numerator and denominator approaches ∞, which doesn't make sense. We can, however, simplify this to analyze the end behavior as follows.

Since the degree of the numerator is less than that of the denominator, therefore we modify the expression by dividing every term in both the numerator and denominator by the greatest power of x in the denominator. By doing so, we'll be able to identify the term in the overall expression that dominates the behavior of the function for large values of x. That is,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2 - 1}{x^3 - 1} &= \lim_{x\to\infty} \frac{\dfrac{x^2 - 1}{x^3}}{\dfrac{x^3 - 1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{x^2}{x^3} - \dfrac{1}{x^3}}{\dfrac{x^3}{x^3} - \dfrac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x} - \dfrac{1}{x^3}}{1 - \dfrac{1}{x^3}} \\
&= \frac{\lim\limits_{x\to\infty} \left(\dfrac{1}{x}\right) - \lim\limits_{x\to\infty} \left(\dfrac{1}{x^3}\right)}{1 - \lim\limits_{x\to\infty} \left(\dfrac{1}{x^3}\right)} \\
&= \frac{0 - 0}{1 - 0} \\
&= 0
\end{aligned}$$

This can be verified by looking at the graph of $f(x) = \dfrac{x^2 - 1}{x^3 - 1}$ as $x \to \infty$ as follows.

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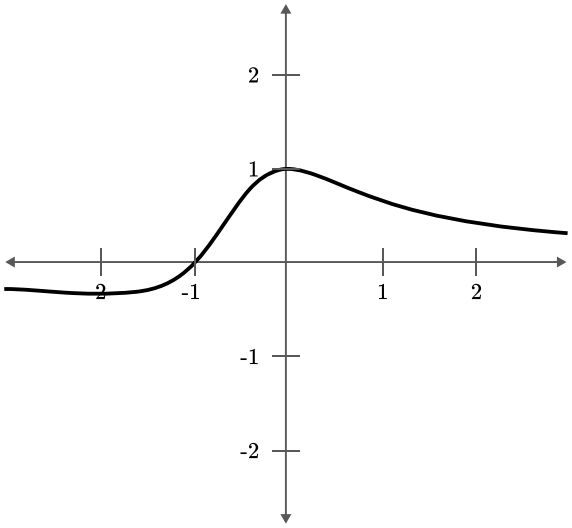

Figure 1.14

Notice that, since $\lim_{x\to\infty} \dfrac{x^2-1}{x^3-1} = 0$, therefore $f(x)$ has an horizontal asymptote at $y = 0$.

Now, we shall consider a function where the degree of the numerator is equal to that of the denominator and $x \to \infty$. That is,

$$f(x) = \frac{9x+\pi}{3x+e}$$

Since π and e are constants, therefore as $x \to \infty$, both the numerator and denominator approach ∞. To deal with this absurdity, we divide each term in both the numerator and denominator by the highest power of x in the denominator as before. That is,

$$\begin{aligned} f(x) &= \frac{9x+\pi}{3x+e} \\ &= \frac{\dfrac{9x+\pi}{x}}{\dfrac{3x+e}{x}} \\ &= \frac{\dfrac{9x}{x}+\dfrac{\pi}{x}}{\dfrac{3x}{x}+\dfrac{e}{x}} \\ &= \frac{9+\dfrac{\pi}{x}}{3+\dfrac{e}{x}} \end{aligned}$$

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$$\therefore f(x) = \frac{9 + \frac{\pi}{x}}{3 + \frac{e}{x}}$$

Now, taking the limit of $f(x)$ as $x \to \infty$, we get,

$$\begin{aligned}\lim_{x\to\infty} \frac{9 + \frac{\pi}{x}}{3 + \frac{e}{x}} &= \frac{9+0}{3+0} \\ &= \frac{9}{3} \\ &= 3\end{aligned}$$

Note that, since $\lim_{x\to\infty} \frac{9x+\pi}{3x+e} = 3$, therefore $f(x)$ has a horizontal asymptote at $y = 3$. This can be seen in the following figure.

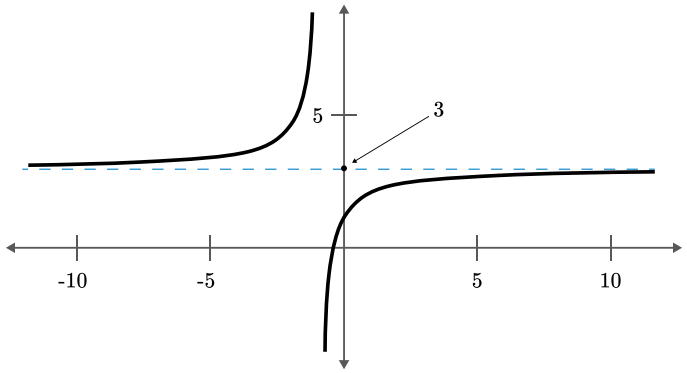

Figure 1.15

Now, having explored rational functions with numerators and denominators of equal degree, as well as those where the numerator's degree is less than that of the denominator, we now focus on rational functions where the numerator's degree exceeds the degree of the denominator. For that, consider the function below,

$$f(x) = \frac{x^2}{1+x}$$

As $x \to \infty$, both the numerator and denominator approaches ∞. To deal with that, we again divide each term in both the numerator and denominator by the highest power of x in the denominator, as before. That is,

$$\begin{aligned}f(x) &= \frac{x^2}{1+x} \\ &= \frac{\frac{x^2}{x}}{\frac{1}{x}+1}\end{aligned}$$

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$$= \frac{x}{1 + \frac{1}{x}}$$
$$\therefore f(x) = \frac{x}{1 + \frac{1}{x}}$$

Now, if we take the limit of $f(x)$ as $x \to \pm\infty$, then the denominator of $f(x)$ approaches the value 1, and the numerator approaches $+\infty$ as $x \to +\infty$ and $-\infty$ as $x \to -\infty$. Thus, the limit is,

$$\lim_{x \to \pm\infty} \frac{x}{1 + \frac{1}{x}} = \pm\infty$$

This presents a special case. Previously, we observed that rational functions tend to approach a horizontal asymptote as $x \to \pm\infty$. However, in this case, the limit doesn't approach a horizontal asymptote as $x \to \pm\infty$. Nevertheless, it does approach a linear asymptote, which becomes apparent when seen visually as follows.

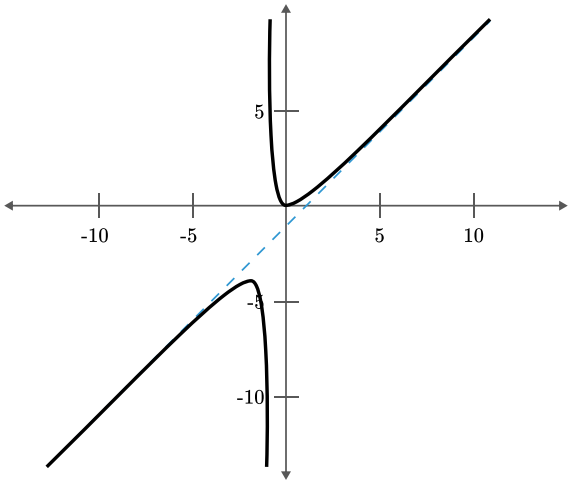

Figure 1.16

Looking at the figure above, it appears that the function $f(x)$ resembles a linear function as $x \to \pm\infty$. To determine the linear function that $f(x)$ is approaching, we do the long division on $f(x)$ and obtain the following.

$$f(x) = \frac{x^2}{1 + x}$$
$$= x - 1 + \frac{1}{x + 1} \qquad \text{by long division}$$

Now, we take the limit as $x \to \pm\infty$,

$$\lim_{x \to \pm\infty} \left(x - 1 + \frac{1}{x + 1} \right) = \lim_{x \to \pm\infty} (x - 1) + \lim_{x \to \pm\infty} \left(\frac{1}{x + 1} \right)$$

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Here, as $x \to \pm\infty$, the term $\dfrac{1}{x+1} \to 0$, therefore,

$$\lim_{x\to\pm\infty}\left(x-1+\frac{1}{x+1}\right) = \lim_{x\to\pm\infty}(x-1)$$

This suggests that the function $f(x)$ behaves similarly to the linear function $x-1$, as $x \to \pm\infty$. For this reason, the graph of $f(x)$ approaches the graph of linear function $x-1$, as $x \to \pm\infty$. Thus, the line $x-1$ is an asymptote of the function $f(x)$ as $x \to \pm\infty$. Such an asymptote is called *slant asymptote* or *oblique asymptote*. Oblique asymptotes occur when the numerator has a degree that is exactly one more than that of the denominator.

In general, when a linear asymptote is neither vertical nor horizontal, it is called an oblique asymptote or slant asymptote. If the distance between a line $mx+b$ and a function $f(x)$ approaches 0, as x increases or decreases indefinitely, that is,

$$\lim_{x\to\pm\infty}\Big(f(x)-(mx+b)\Big) = 0$$

then the line $mx+b$ is called the oblique asymptote.

Summarizing these function's behavior, we have the following remark.

Remark 1.5.3.
Let $P(x)$ and $Q(x)$ be polynomial functions such that

$$\begin{aligned} R(x) &= \frac{P(x)}{Q(x)} \\ &= \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} \end{aligned}$$

where, $a_n \neq 0$, $b_m \neq 0$. Now,

1. If the degrees of the numerator n and denominator m are equal (i.e., $n = m$), then $R(x)$ has a horizontal asymptote at $y = \dfrac{a_n}{b_m}$, as $x \to \pm\infty$.

2. If the degree of the numerator n is less than the denominator m (i.e., $n < m$), then $R(x)$ has a horizontal asymptote at $y = 0$, as $x \to \pm\infty$.

3. If the degree of the numerator n is greater than the denominator m (i.e., $n > m$), then $R(x)$ does not have neither vertical nor horizontal asymptote, as $x \to \pm\infty$.

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A few examples are shown below.

► **Example 1.5.11.** Evaluate $\lim\limits_{x\to\pm\infty} \dfrac{2x^3+3x^2+5x+7}{11x^2+13x+17}$

Solution. Since the degree of the numerator (that is, 3) is greater than the degree of the denominator (that is, 2), we divide the numerator and denominator by x^n, where n is the greatest power in the denominator (that is, 2).

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x^3+3x^2+5x+7}{11x^2+13x+17} &= \lim_{x\to\pm\infty} \frac{\dfrac{2x^3+3x^2+5x+7}{x^2}}{\dfrac{11x^2+13x+17}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{\dfrac{2x^3}{x^2}+\dfrac{3x^2}{x^2}+\dfrac{5x}{x^2}+\dfrac{7}{x^2}}{\dfrac{11x^2}{x^2}+\dfrac{13x}{x^2}+\dfrac{17}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{2x+3+\dfrac{5}{x}+\dfrac{7}{x^2}}{11+\dfrac{13}{x}+\dfrac{17}{x^2}} \\
&= \pm\infty
\end{aligned}$$

Therefore, as $x\to+\infty$, the function $f(x)=\dfrac{2x^3+3x^2+5x+7}{11x^2+13x+17}\to+\infty$. Similarly, as $x\to-\infty$, the function $f(x)=\dfrac{2x^3+3x^2+5x+7}{11x^2+13x+17}\to-\infty$. ∎

► **Example 1.5.12.** Evaluate $\lim\limits_{x\to\pm\infty} \dfrac{\pi x^3+e}{ex^2+\pi}$

Solution. Since the degree of the numerator (that is, 3) is greater than the degree of the denominator (that is, 2), we divide the numerator and denominator by x^n, where n is the greatest power in the denominator.

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\pi x^3+e}{ex^2+\pi} &= \lim_{x\to\pm\infty} \frac{\dfrac{\pi x^3+e}{x^2}}{\dfrac{ex^2+\pi}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{\dfrac{\pi x^3}{x^2}+\dfrac{e}{x^2}}{\dfrac{ex^2}{x^2}+\dfrac{\pi}{x^2}}
\end{aligned}$$

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$$= \lim_{x\to\pm\infty} \frac{\pi x + \dfrac{e}{x^2}}{e + \dfrac{\pi}{x^2}}$$

$$= \pm\infty$$

Therefore, as $x \to +\infty$, the function $f(x) = \dfrac{\pi x^3 + e}{ex^2 + \pi} \to +\infty$. Similarly, as $x \to -\infty$, the function $f(x) = \dfrac{\pi x^3 + e}{ex^2 + \pi} \to -\infty$. ∎

► **Example 1.5.13.** Evaluate $\lim\limits_{x\to\pm\infty} \dfrac{3x^3 + 2x^2}{9x^3 + 6}$

Solution. Since the degree of the numerator (that is, 3) is equal to the degree of the denominator (that is also 3), we divide the numerator and denominator by x^n, where n is the greatest power in the denominator.

$$\lim_{x\to\pm\infty} \frac{3x^3 + 2x^2}{9x^3 + 6} = \lim_{x\to\pm\infty} \frac{\dfrac{3x^3 + 2x^2}{x^3}}{\dfrac{9x^3 + 6}{x^3}}$$

$$= \lim_{x\to\pm\infty} \frac{\dfrac{3x^3}{x^3} + \dfrac{2x^2}{x^3}}{\dfrac{9x^3}{x^3} + \dfrac{6}{x^3}}$$

$$= \lim_{x\to\pm\infty} \frac{3 + \dfrac{2}{x}}{9 + \dfrac{6}{x^3}}$$

$$= \frac{3}{9}$$

Therefore, as $x \to +\infty$, the function $f(x) = \dfrac{3x^3 + 2x^2}{9x^3 + 6} \to \dfrac{3}{9}$. Similarly, as $x \to -\infty$, the function $f(x) = \dfrac{3x^3 + 2x^2}{9x^3 + 6} \to \dfrac{3}{9}$. ∎

► **Example 1.5.14.** Evaluate $\lim\limits_{x\to\pm\infty} \dfrac{2e\pi x^2 + x}{x^2 + 2x}$

Solution. Since the degree of the numerator (2) is equal to the degree of the denominator (2), divide the numerator and denominator by x^n, where n is the

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greatest power in the denominator. That is,

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2e\pi x^2 + x}{x^2 + 2x} &= \lim_{x\to\pm\infty} \frac{\dfrac{2e\pi x^2 + x}{x^2}}{\dfrac{x^2 + 2x}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{\dfrac{2e\pi x^2}{x^2} + \dfrac{x}{x^2}}{\dfrac{x^2}{x^2} + \dfrac{2x}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{2e\pi + \dfrac{1}{x}}{1 + \dfrac{2}{x}} \\
&= 2e\pi
\end{aligned}$$

Therefore, as $x \to +\infty$, the function $f(x) = \dfrac{2e\pi x^2 + x}{x^2 + 2x} \to 2e\pi$. Similarly, as $x \to -\infty$, the function $f(x) = \dfrac{2e\pi x^2 + x}{x^2 + 2x} \to 2e\pi$. ∎

► **Example 1.5.15.** Evaluate $\lim_{x\to\pm\infty} \dfrac{3x^3 + x}{5x^5 + 2x^2}$

Solution. Since the degree of the numerator (3) is less than the degree of the denominator (5), we divide the numerator and denominator by x^n, where n is the greatest power in the denominator. That is,

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{3x^3 + x}{5x^5 + 2x^2} &= \lim_{x\to\pm\infty} \frac{\dfrac{3x^3 + x}{x^5}}{\dfrac{5x^5 + 2x^2}{x^5}} \\
&= \lim_{x\to\pm\infty} \frac{\dfrac{3x^3}{x^5} + \dfrac{x}{x^5}}{\dfrac{5x^5}{x^5} + \dfrac{2x^2}{x^5}} \\
&= \lim_{x\to\pm\infty} \frac{\dfrac{3}{x^2} + \dfrac{1}{x^4}}{5 + \dfrac{2}{x^3}} \\
&= \frac{0}{5} \\
&= 0
\end{aligned}$$

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Therefore, as $x \to +\infty$, the function $f(x) = \dfrac{3x^3 + x}{5x^5 + 2x^2} \to 0$. Similarly, as $x \to -\infty$, the function $f(x) = \dfrac{3x^3 + x}{5x^5 + 2x^2} \to 0$. ∎

► **Example 1.5.16.** Evaluate $\lim\limits_{x \to \pm\infty} \dfrac{x^3 + 2x^2 + 3x + 4}{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}$

Solution. Since the degree of the numerator (3) is less than the degree of the denominator (5), we divide the numerator and denominator by x^n, where n is the greatest power in the denominator. That is,

$$\begin{aligned}
\lim_{x \to \pm\infty} \frac{x^3 + 2x^2 + 3x + 4}{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6} &= \lim_{x \to \pm\infty} \frac{\dfrac{x^3 + 2x^2 + 3x + 4}{x^5}}{\dfrac{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}{x^5}} \\
&= \lim_{x \to \pm\infty} \frac{\dfrac{x^3}{x^5} + \dfrac{2x^2}{x^5} + \dfrac{3x}{x^5} + \dfrac{4}{x^5}}{\dfrac{x^5}{x^5} + \dfrac{2x^4}{x^5} + \dfrac{3x^3}{x^5} + \dfrac{4x^2}{x^5} + \dfrac{5x}{x^5} + \dfrac{6}{x^5}} \\
&= \lim_{x \to \pm\infty} \frac{\dfrac{1}{x^2} + \dfrac{2}{x^3} + \dfrac{3}{x^4} + \dfrac{4}{x^5}}{1 + \dfrac{2}{x} + \dfrac{3}{x^2} + \dfrac{4}{x^3} + \dfrac{5}{x^4} + \dfrac{6}{x^5}} \\
&= \frac{0}{1} \\
&= 0
\end{aligned}$$

Therefore, as $x \to +\infty$, the function $f(x) = \dfrac{x^3 + 2x^2 + 3x + 4}{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6} \to 0$. Similarly, as $x \to -\infty$, the function $f(x) = \dfrac{x^3 + 2x^2 + 3x + 4}{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6} \to 0$. ∎

· · · / Subsec. 1.5.3 / Segment 1.5.3.4 : The Limit of Functions Involving Radicals as $x \to \pm\infty$

Previously, we've seen how to deal with the limit of rational functions as x increases or decreases indefinitely. However, sometimes these functions can contain a radical. For example, consider the same function we previously examined

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as $x \to \infty$, but this time the numerator contains a radical.

$$f(x) = \frac{\sqrt{9x+\pi}}{3x+e}$$

Now, to evaluate the limit of this function as $x \to \infty$, we divide the numerator and denominator by the highest power of x. That is,

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{9x+\pi}}{3x+e} &= \lim_{x\to\infty} \frac{\dfrac{\sqrt{9x+\pi}}{x}}{\dfrac{3x+e}{x}} \\ &= \lim_{x\to\infty} \frac{\dfrac{\sqrt{9x+\pi}}{x}}{3+\dfrac{e}{x}}\end{aligned}$$

Now, a problem arises due to the presence of a radical in the numerator. To address this issue, notice that $x \geq 0$ as $x \to +\infty$, therefore

$$x = |x| = \sqrt{x^2}$$

. Therefore, it is totally possible for us to evaluate the limit in the following manner.

$$\begin{aligned}\lim_{x\to\infty} \frac{\dfrac{\sqrt{9x+\pi}}{x}}{3+\dfrac{e}{x}} &= \lim_{x\to\infty} \frac{\dfrac{\sqrt{9x+\pi}}{\sqrt{x^2}}}{3+\dfrac{e}{x}} \\ &= \lim_{x\to\infty} \frac{\sqrt{\dfrac{9x+\pi}{x^2}}}{3+\dfrac{e}{x}} \\ &= \lim_{x\to\infty} \frac{\sqrt{\dfrac{9}{x}+\dfrac{\pi}{x^2}}}{3+\dfrac{e}{x}} \\ &= \frac{\sqrt{0+0}}{3+0} \\ &= 0\end{aligned}$$

This same strategy can be used to evaluate the limit of the following function. Consider the function below.

$$f(x) = \frac{10x^2+x}{\sqrt{2x^2+8}}$$

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What is the limit of $f(x)$ as $x \to +\infty$ and $x \to -\infty$?

We will use the previous strategy. Divide the numerator and denominator by the highest power of x in the denominator. The highest power of x in the denominator of $f(x)$ is x^2. However, since x^2 is under a square root, therefore we need to divide both the numerator and denominator by $\sqrt{x^2}$. That is,

$$\begin{aligned}
\lim_{x\to+\infty} \frac{10x^2+x}{\sqrt{2x^2+8}} &= \lim_{x\to+\infty} \frac{\dfrac{10x^2+x}{\sqrt{x^2}}}{\dfrac{\sqrt{2x^2+8}}{\sqrt{x^2}}} \\
&= \lim_{x\to+\infty} \frac{\dfrac{10x^2+x}{\sqrt{x^2}}}{\sqrt{\dfrac{2x^2+8}{x^2}}} \\
&= \lim_{x\to+\infty} \frac{\dfrac{10x^2}{\sqrt{x^2}}+\dfrac{x}{\sqrt{x^2}}}{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{8}{x^2}}}
\end{aligned}$$

Since $\sqrt{x^2} = |x| = x$, for all $x \geq 0$, therefore,

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\dfrac{10x^2}{\sqrt{x^2}}+\dfrac{x}{\sqrt{x^2}}}{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{8}{x^2}}} &= \lim_{x\to+\infty} \frac{\dfrac{10x^2}{x}+\dfrac{x}{x}}{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{8}{x^2}}} \\
&= \lim_{x\to+\infty} \frac{10x+1}{\sqrt{2+\dfrac{8}{x^2}}} \\
&= \frac{+\infty+1}{\sqrt{2+0}} \\
&= +\infty
\end{aligned}$$

Therefore, $f(x) \to +\infty$, as $x \to +\infty$.

Let us now consider $x \to -\infty$, where $\sqrt{x^2} = |x| = -x$, for all $x < 0$,

$$\lim_{x\to-\infty} \frac{10x^2+x}{\sqrt{2x^2+8}} = \lim_{x\to-\infty} \frac{\dfrac{10x^2+x}{\sqrt{x^2}}}{\dfrac{\sqrt{2x^2+8}}{\sqrt{x^2}}}$$

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$$
\begin{aligned}
&= \lim_{x\to-\infty} \frac{\dfrac{10x^2+x}{-x}}{\sqrt{\dfrac{2x^2+8}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{\dfrac{10x^2}{-x}+\dfrac{x}{-x}}{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{8}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{-10x-1}{\sqrt{2+\dfrac{8}{x^2}}} \\
&= \frac{-(-\infty)-1}{\sqrt{2+0}} \\
&= +\infty
\end{aligned}
$$

Therefore, $f(x) \to +\infty$, as $x \to \pm\infty$.

⋯ / Subsec. 1.5.3 / Segment 1.5.3.5 : Limits of Transcendental Functions as $x \to \pm\infty$

Transcendental functions, such as exponential, logarithmic, and trigonometric functions, behave in distinct ways as $x \to \infty$ or $x \to -\infty$.

Remark 1.5.4. (Exponential Functions) Let $f(x) = a^x$, where $a > 1$, then,

$$\lim_{x\to\infty} a^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} a^x = 0$$

If $0 < a < 1$, then,

$$\lim_{x\to\infty} a^x = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} a^x = \infty$$

A few examples are provided below.

► **Example 1.5.17.** Evaluate $\lim_{x\to\infty} e^x$

Solution. Since $e > 1$, therefore, in this case, exponential functions increase without a bound as $x \to \infty$. Thus,

$$\lim_{x\to\infty} e^x = \infty$$

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▮

► **Example 1.5.18.** Evaluate $\lim_{x\to-\infty} \pi^x$

Solution. Since $\pi > 1$, therefore π^x approach zero as $x \to -\infty$. Thus,

$$\lim_{x\to-\infty} \pi^x = 0$$

▮

► **Example 1.5.19.** Evaluate $\lim_{x\to\infty} e^{-x}$

Solution. Since $\lim_{x\to\infty} e^x = \infty$, therefore,

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{e^x} &= \frac{1}{\lim_{x\to\infty} e^x} \\ &= 0\end{aligned}$$

Therefore, $\lim_{x\to\infty} e^{-x} = 0$ ▮

Remark 1.5.5. (Logarithmic Functions) Let $f(x) = \log_a(x)$, where $a > 1$, then,

$$\lim_{x\to\infty} \log_a(x) = \infty \qquad \text{and} \qquad \lim_{x\to0^+} \log_a(x) = \infty$$

If $0 < a < 1$, then,

$$\lim_{x\to\infty} \log_a(x) = -\infty \qquad \text{and} \qquad \lim_{x\to0^+} \log_a(x) = \infty$$

A few examples are provided below.

► **Example 1.5.20.** Evaluate $\lim_{x\to\infty} \ln(x)$

Solution. Since $\ln(x) = \log_e(x)$ and $e > 1$, therefore, the natural logarithmic function grows slowly but unbounded as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \ln(x) = \infty$$

▮

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► **Example 1.5.21.** Evaluate $\lim_{x\to 0^+} \ln(x)$

Solution. Since $\ln(x) = \log_e(x)$ and $e > 1$, therefore, the natural logarithmic function approaches negative infinity as x approaches 0 from the positive side. Thus,

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

∎

Remark 1.5.6. (Trigonometric Functions) The behavior of six trigonometric functions as $x \to \pm\infty$ are as follows.

1. Sine Function: As $x \to \pm\infty$, $f(x) = \sin(x)$ does not approach a single value; instead, it oscillates between -1 and 1 indefinitely. Therefore,
$$\lim_{x\to\pm\infty} \sin(x) \qquad \text{does not exist}$$
2. Cosine Function: As $x \to \pm\infty$, $f(x) = \cos(x)$ does not approach a single value; instead, it oscillates between -1 and 1 indefinitely. Therefore,
$$\lim_{x\to\pm\infty} \cos(x) \qquad \text{does not exist}$$
3. Tangent Function: The tangent function $f(x) = \tan(x)$ has vertical asymptotes at $x = \dfrac{(2n+1)\pi}{2}$, where $n \in \mathbb{Z}$. However, between these asymptotes, $\tan(x)$ increases or decreases without a bound. Therefore,
$$\lim_{x\to\pm\infty} \tan(x) \qquad \text{does not exist}$$
4. Cotangent Function: The Cotangent function $f(x) = \cot(x)$ has vertical asymptotes at $x = n\pi$, where $n \in \mathbb{Z}$. However, between these asymptotes, $\cot(x)$ increases or decreases without a bound. Therefore,
$$\lim_{x\to\pm\infty} \cot(x) \qquad \text{does not exist}$$
5. Secant Function: The Secant function $f(x) = \sec(x)$ has vertical asymptotes at $x = \dfrac{(2n+1)\pi}{2}$, where $n \in \mathbb{Z}$. However, between these

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asymptotes, $\sec(x)$ increases or decreases without a bound. Therefore,

$$\lim_{x\to\pm\infty} \sec(x) \qquad \text{does not exist}$$

6. Cosecant Function: The Cosecant function $f(x) = \csc(x)$ has vertical asymptotes at $x = n\pi$, where $n \in \mathbb{Z}$. However, between these asymptotes, $\csc(x)$ increases or decreases without a bound. Therefore,

$$\lim_{x\to\pm\infty} \csc(x) \qquad \text{does not exist}$$

———————The End of Section 1.5

Limits Involving ∞

§ 1.6 The Concept of Continuity

The concept of continuity is one of the most important concepts in the whole of mathematics. It describes the behavior of a function. A function $f(x)$ is said to be continuous if its graph has no interruptions and maintains a smoothness across every point. An analogy can be drawn to railway tracks, where an uninterrupted and smooth track is essential for safe travel. Likewise, for a function to be considered well-behaved, it must demonstrate continuity across all points.

In informal terms, a function is considered continuous if its graph can be drawn without lifting the pen from the page.

We begin by discussing the concept of continuity for a function $f(x)$ at a specific point $x = a$. Following that, we'll delve into functions that exhibit continuity across an interval.

Chap. 1 / Sec. 1.6 / Subsec. 1.6.1 : Continuity at a Point

A function $f(x)$ is continuous at a point $x = a$, if it does not have any discontinuities. Discontinuities are points at which a function does not behave "smoothly" or continuously. Discontinuities are classified into three types: removable discontinuities, jump discontinuities, and infinite discontinuities. Each has distinct characteristics that impact how functions behave at a certain point.

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1. **Removable Discontinuities:** A removable discontinuity occurs when a function has a "hole" at a certain point, but can be made continuous by appropriately redefining the function at that point. Essentially, the limit of the function as it approach the point of discontinuity exists, but the function's value either is not defined or differs from this limit. For example, consider the following figures.

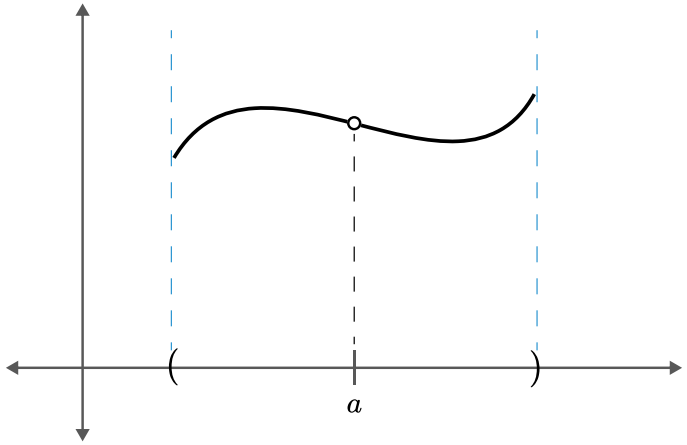

Figure 1.17

Here, the function $f(x)$ is not defined at the point $x = a$, therefore $f(x)$ is discontinuous at the point $x = a$. A similarity can be seen in the following figure.

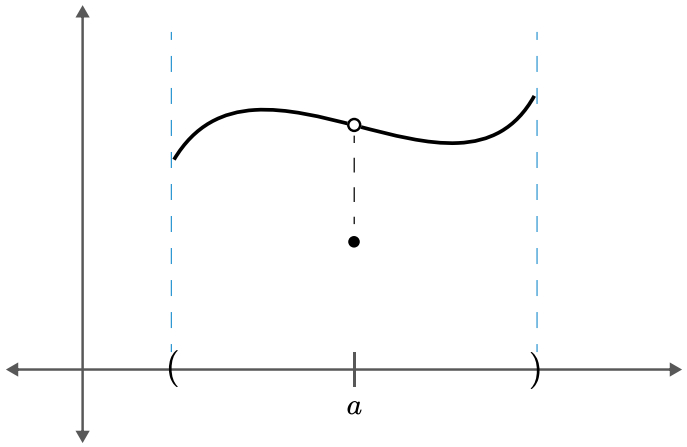

Figure 1.18

Here, the value of the function $f(x)$ at the point $x = a$ differ from the limit of $f(x)$ as $x \to a$, therefore $f(x)$ is discontinuous at the point $x = a$.

2. **Jump Discontinuities:** A jump discontinuity occurs when there is a

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sudden "jump" in the value of the function at a certain point, causing the left-sided and the right-sided limit to differ. The function's value jumps from one level to another without passing through any intermediate values. For example, consider the figure below.

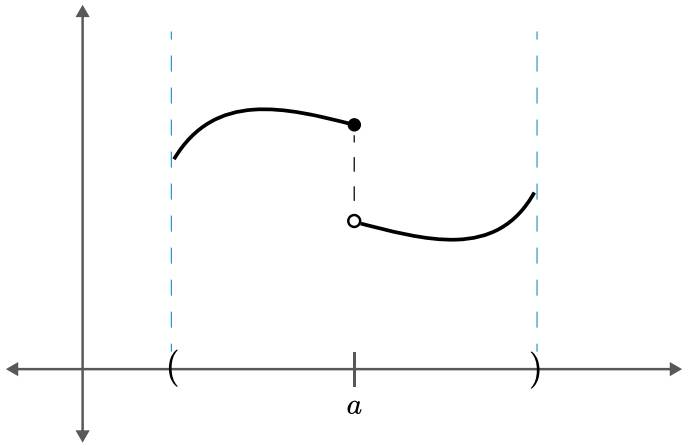

Figure 1.19

Here, $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, therefore the limit of $f(x)$ as $x \to a$ does not exist.

3. **Infinite Discontinuities**: An infinite discontinuity occurs when the value of the function $f(x)$ approaches infinity as x approaches the point of discontinuity. Rational functions often have discontinuities at points where the denominator is zero, resulting in vertical asymptotes. For example, consider the figure below.

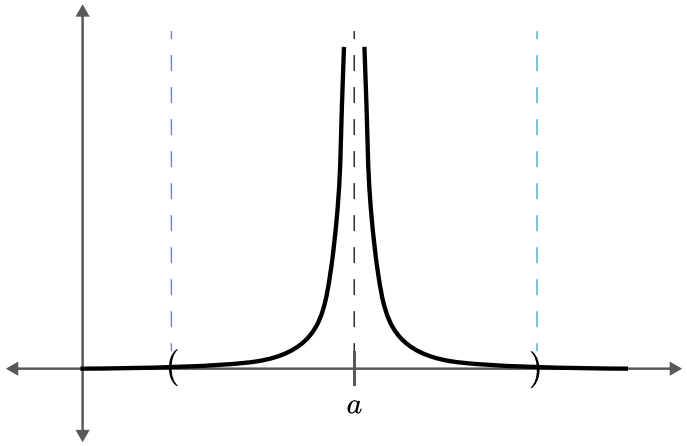

Figure 1.20

Here, as $x \to a$, the limit of the function $f(x)$ grows indefinitely, therefore

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the limit of $f(x)$ as $x \to a$ has an infinite discontinuity.

Having seen a few descriptions of discontinuities, we now summarize to define the continuity of a function at a point as follows.

Definition 1.6.1. (Continuity at a Point) A function $f(x)$ is continuous at a point a, if

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

If a function does not satisfy any of the specified criteria at the point a, then the function is not continuous at that point.

Chap. 1 / Sec. 1.6 / Subsec. 1.6.2 : Continuity over an Interval

When we talk about a function being continuous over an interval, we are saying that the function behaves predictably and without interruption across all points within that interval. In other words, the graph of the function can be traced without lifting the pen. In order to fully understand continuity over an interval, it's essential to first recognize the type of intervals.

1. **Open intervals**: An interval (a, b) is called open if it includes all points between a and b, but not the endpoints a and b themselves. A function is continuous on an open interval if it satisfies the conditions of continuity at a point, across all the interior points.

2. **Closed intervals**: An interval $[a, b]$ is called closed if it includes all points between a and b, as well as the endpoints a and b themselves. When defining continuity at a point, we rely on two-sided limits, which consider both directions around a point. However, at the endpoints of a closed interval, two-sided limits aren't possible since there's no interval extending beyond that boundary. To handle this, we define continuity at an endpoint using a one-sided limit. In other words, a function is continuous at an endpoint if the one-sided limit exists and equals the function's value at that point.

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3. **Half-open intervals**: If we combine the open and the closed intervals together, we get intervals such as $(a, b]$ or $[a, b)$. These intervals are semi-closed, that is, they include only one of the endpoints and not the other. Continuity over a half-open interval requires that the function be continuous at all interior points as well as at the endpoint that is included.

To understand this concept visually, consider the following figure.

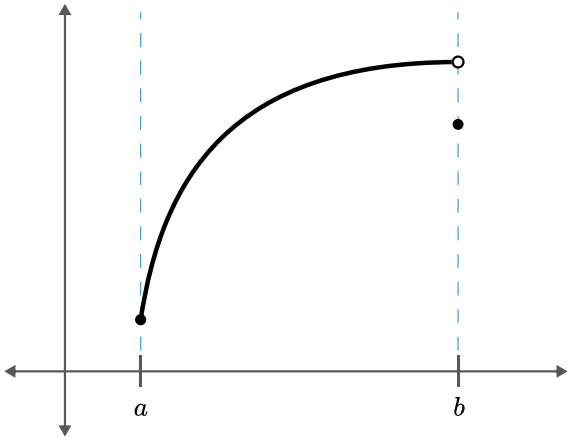

Figure 1.21

In the figure above, the function $f(x)$ is continuous over the open interval (a, b) and at the *left* endpoint over the closed interval $[a, b]$, because

$$\lim_{x \to a^+} f\left(x\right) = f\left(a\right)$$

However, the function $f(x)$ is *not* continuous at the *right* endpoint over the closed interval $[a, b]$, because

$$\lim_{x \to b^-} f\left(x\right) \neq f\left(b\right)$$

In general, we have the following definition for continuity at endpoints.

Definition 1.6.2. (Continuity at Endpoints)

1. Left-continuous: A function $f(x)$ is continuous from the left at the point a, if
$$\lim_{x \to a^-} f\left(x\right) = f\left(a\right)$$
.

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2. Right-continuous: A function $f(x)$ is continuous from the right at the point a, if
$$\lim_{x \to a^+} f(x) = f(a)$$
.

Now, based on definition above, the continuity of a function over a closed interval $[a, b]$ is defined as follows.

Definition 1.6.3. (Continuity over an Closed Interval)
A function $f(x)$ is continuous over a closed interval $[a, b]$, if the following is true:

1. $f(x)$ is continuous over an open interval (a, b).
2. $\lim_{x \to a^+} f(x) = f(a)$
3. $\lim_{x \to b^-} f(x) = f(b)$

To fully grasp the concept of continuity over an interval, consider the following example.

► **Example 1.6.1.** Consider the figure below to identify points where $f(x)$ has discontinuities within the interval $[0, 8]$.

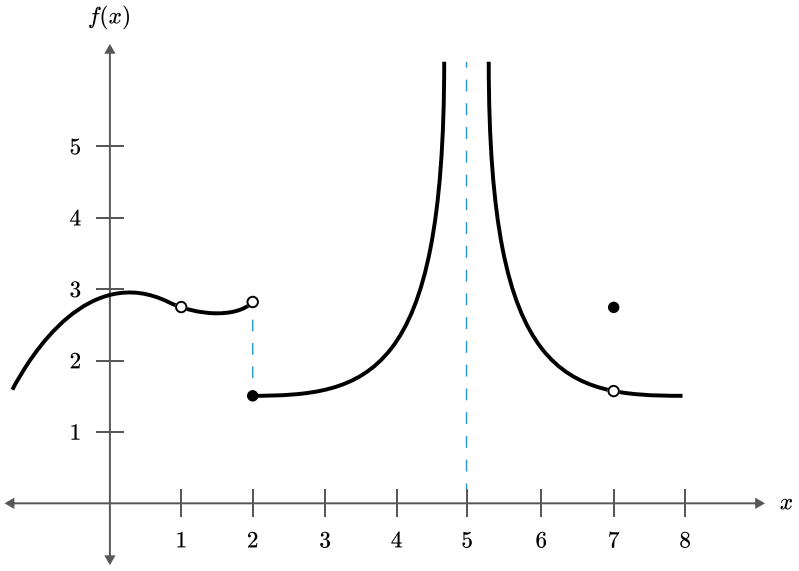

Figure 1.22

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Solution. Using the definition of continuity of a function, we get the following.

1. $f(x)$ is discontinuous at $x = 1$ because $f(1)$ is not defined.
2. $f(x)$ is discontinuous at $x = 2$ because $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$
3. $f(x)$ is discontinuous at $x = 5$ because $\lim\limits_{x\to 3} f(x) = \infty$
4. $f(x)$ is discontinuous at $x = 7$ because $\lim\limits_{x\to 4} f(x) \neq f(4)$

Note that the discontinuities of $f(x)$ at points $x = 1, 7$ are called removable discontinuity as they can be removed by redefining the function at these points. The discontinuity at $x = 2$ is called jump discontinuity. Whereas, the discontinuity at $x = 5$ is called infinite discontinuity. ∎

Chap. 1 / Sec. 1.6 / Subsec. 1.6.3 : Properties of Continuity

In the previous subsection, we discussed the concept of a function's continuity. However, when determining the continuity of a complex function, it is often helpful to break the function down into simpler functions. The idea is that if the complex function can be expressed in terms of simpler functions that are continuous, then the entire function will also be continuous. For example, if the complex function is written as a sum, difference, product, quotient (with a non-zero denominator), or composition of continuous functions, then the whole function remains continuous. This approach simplifies the analysis by allowing us to assess continuity through its simpler components, rather than tackling the entire function at once. We discuss this matter in details in the segments below.

· · · / Subsec. 1.6.3 / Segment 1.6.3.1 : Continuity of Combined Functions

Functions can be combined in various ways, such as through addition, subtraction, multiplication, division, or composition. Understanding the continuity of these combined functions is essential because it allows us to determine how the continuity of individual functions influences the continuity of the overall function. Fortunately, there are clear rules that dictate when combined functions remain continuous, based on the continuity of their component functions as summarized below.

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Theorem 1.6.1. (Continuity of Combined Functions)
If c is a real number and n is a positive integer, and both functions $f(x)$ and $g(x)$ are continuous at the point a, then the functions,

1. $c \cdot f(x)$
2. $f(x) + g(x)$
3. $f(x) - g(x)$
4. $f(x) \cdot g(x)$
5. $\dfrac{f(x)}{g(x)}$, where, $g(x) \neq 0$

are also continuous at the point a.

The following types of functions are continuous at every point within their respective domains.

Remark 1.6.1. (Continuous Functions)
The functions below are continuous at every point within their defined domains.

1. **Polynomial Functions**, that is, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$
2. **Rational Functions**, that is, $R(x) = \dfrac{P(x)}{Q(x)}$, where, $P(x)$ and $Q(x)$ are both polynomials and $Q(x) \neq 0$.
3. **Radical Functions**, that is, $f(x) = \sqrt[n]{x}$; provided that it is defined
4. **Trigonometric Functions**, that is, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$
5. **Inverse Trigonometric Functions**, that is, $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, $\cot^{-1}(x)$, $\sec^{-1}(x)$, $\csc^{-1}(x)$.
6. **Exponential and Logarithmic Functions**, that is, $f(x) = a^x$, $f(x) = e^x$, $f(x) = \ln(x)$, $f(x) = \log_b(x)$

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· · · / Subsec. 1.6.3 / Segment 1.6.3.2 : Continuity of Composite Functions

A composite function $f(g(x))$ is a result of applying the function g to the input x and then applying the function f to the output of $g(x)$. To determine whether the composite function is continuous, we rely on the following theorem.

Theorem 1.6.2. (Continuity of a Composite Function) If $g(x)$ is continuous at the point a and f is continuous at the point $g(a)$, then the composite function, $f(g(x))$ is also continuous at the point a.

The theorem above suggests that for $f(g(x))$ to be continuous at a point a, both the inner function $g(x)$ must be continuous at a, as well as the outer function f must be continuous at the value that $g(x)$ takes at a, that is, $g(a)$. For example, consider $f(g(x)) = \sin(x^2)$, where $f(x) = \sin(x)$ and $g(x) = x^2$. The function $g(x) = x^2$ is continuous on its entire domain, and the function $f(x) = \sin(x)$ is also continuous everywhere. Therefore, the function $f(g(x)) = \sin(x^2)$ is also continuous everywhere.

Chap. 1 / Sec. 1.6 / Subsec. 1.6.4 : The Intermediate Value Theorem (IVT)

The Intermediate Value Theorem (IVT) is a fundamental result in Calculus. It formalizes the intuitive idea that if a function is continuous on a closed interval, and if it takes on two distinct values at the endpoints of that interval, then it must take on every value between those two endpoints at some point within the interval. The idea is indeed intuitive and to get a gist of it, consider the following analogy.

Suppose we're hiking up a hill, starting at point A on the ground and heading upwards to point B, which is at a higher elevation. The path we're taking is smooth, without any sudden drops or jumps. As we walk up, we inevitably pass through every height between A and B. For instance, if we start at 0 meters and end at 100 meters, then there must be a point where we have been at 50 meters. This is because the path is continuous, so we can't skip any heights or jump straight from 0 to 100 meters. This is the core idea of the Intermediate Value Theorem.

Suppose now the hill's height is represented by a continuous function, $f(x)$, and

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let two points a and b on the x-axis represent the start and end of the walk, respectively. If $f(a) = 0$ and $f(b) = 100$, then there must be a point c between a and b such that the function's value $f(c) = 50$. This is guaranteed by the continuity of the function.

Following the same logic as in the analogy above, we can state the following theorem.

Theorem 1.6.3. (The Intermediate Value Theorem (IVT))
If $f(x)$ is a continuous function defined on the interval $[a, b]$ and L is any number between $f(a)$ and $f(b)$. Then, there exists at least one point $c \in (a, b)$ such that,

$$f(c) = L$$

The Intermediate Value Theorem (IVT) can be used in many clever ways to ensure the existence of solutions within a certain interval. A few of them are shown below.

► **Example 1.6.2.** Prove that the function $f(x) = x^3 - x - 2$ has a solution on the interval $[1, 2]$.

Solution. Notice that $f(x) = x^3 - x - 2$ is a polynomial, and therefore it is continuous everywhere. As a result, it is also continuous on the interval $[1, 2]$.

Now, evaluating $f(x)$ for $x = 1$ yields: $f(1) = 1^3 - 1 - 2 = -2$ Similarly, $f(x)$ for $x = 2$ yields: $f(2) = 2^3 - 2 - 2 = 4$

Since $f(1) = -2$ and $f(2) = 4$, therefore, according to the Intermediate Value Theorem, there exist a point $c \in [1, 2]$ where $f(x)$ changed its sign from negative to positive, resulting 0. In other words, there must be at least one point $c \in [a, b]$ such that $f(c) = 0$. Thus, we have proved that the function $f(x) = x^3 - x - 2$ has a solution at the point c. ∎

► **Example 1.6.3.** Prove that the function $f(x) = x^3 - 10$ has a solution on the interval $[2, 2.2]$.

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Solution. The function $f(x) = x^3 - 10$ is a polynomial, and therefore it is continuous everywhere. That's why we evaluate $f(x)$ at the interval's endpoints to identify a point where the function may have changed its sign.

$$f(2) = 2^3 - 10 = -2$$
$$f(2.2) = 2.2^3 - 10 = 0.648$$

Since $f(2) = -2$ and $f(2.2) = 0.648$, therefore, by the Intermediate Value Theorem, there must exist a point $c \in (2, 2.2)$ such that $f(c) = 0$, because $f(x)$ is continuous and crosses the x-axis between $x = 2$ and $x = 2.2$. ∎

———————The End of Section 1.6
The Concept of Continuity

§ 1.7 ($\varepsilon - \delta$) Definition of Limit

The $(\varepsilon - \delta)$ definition of a limit serves as a pivotal shift in Calculus from intuitive reasoning to formal mathematics. The early development of calculus was marked by controversies and ambiguities regarding the fundamental principles upon which it was built. The intuitive idea that a function $f(x)$ approaches some value L as its input x approaches some value a is always suggestive, but lacks a clear definition of vague words such as "approach."

Indeed, what does it mean for a function to "approach" a value? It may be intuitively clear, but this intuition is not what to be called mathematics.

To understand why, suppose you are the mathematician who devised the concept of limits. Intuitively, what you've written in the following is pretty clear.

$$\lim_{x \to a} f(x) = L$$

That is, as x gets close to a, the function $f(x)$ gets close to L.

Now, the question arises: how do you define what it means to be "close," as you have used this term in your definition of limits? Specifically, what does it mathematically mean for some variable x to be "close" to a number a? After pondering, you may figure out that the distance between x and a is simply $x - a$. Since distance cannot be negative, therefore you use the absolute value of $x - a$,

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that is, $|x-a|$. So, "closeness" is, in this case, if the distance $|x-a|$ is kept small.

So, this implies that the function $f(x)$ is close to a value L if the distance between $f(x)$ and L, that is, $|f(x) - L|$ is kept small by keeping the distance between x and a sufficiently small.

The issue, however, is not resolved. Because, you may have clearly defined what "closeness" implies in this case, but it was done so by leveraging the vagueness of the term "smallness." That is, "smallness" is a relative term, so "how small is really small?" is a question to be answered now.

To address the issue, you can assign a fixed real number to indicate the exact amount of "smallness." Say, $\varepsilon \in \mathbb{R}$ is such a real number. So, $|f(x) - L| < \varepsilon$ implies that the distance between $f(x)$ and L is less than the chosen "small" real number ε, no matter how small. In other words, $f(x)$ is within ε-distance of L.

You can similarly assign a fixed value to clearly define the "smallness" for the distance between x and a. Let $\delta \in \mathbb{R}$ be such a real number. So, $|x - a| < \delta$ implies that the distance between x and a is less than the chosen "small" real number δ, no matter how small.

So, after precisely defining the ambiguous words, you can now ensure that if $|x-a| < \delta$, then $|f(x)-L| < \varepsilon$, where ε is the chosen value. However, you soon realize that since $x \neq a$ for the reason that the function may not even be defined at $x = a$, therefore, the distance between x and a can never be zero, that is, $|x - a| > 0$. So, this implies that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

In other words, to keep the distance between $f(x)$ and L less than ε, i.e., $|f(x) - L| < \varepsilon$; you would have to keep the distance between x and a less than δ and greater than 0, i.e., $0 < |x - a| < \delta$. Notice that the magnitude of δ depends on the function $f(x)$ itself. For very steep functions, you would need a much smaller δ to keep $f(x)$ within a given ε. So, this implies that for every chosen real number $\varepsilon > 0$ (no matter how small), there exists a corresponding $\delta > 0$, but δ will vary based on the behavior of the function.

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So, putting everything together, we write the following

$$\lim_{x \to a} f(x) = L$$

to mean that the limit of $f(x)$, as x approaches a value a, is L; if the following property holds:

for every real number $\varepsilon > 0$, there exist a corresponding real number $\delta > 0$ such that for all real number x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

All the details of the $(\varepsilon - \delta)$ definition might seem overwhelming at once. Therefore, to better illustrate this concept, we will consider a visual approach. For that, consider the following figure.

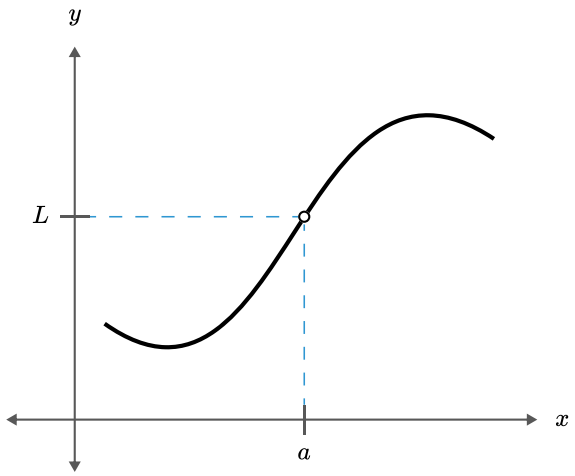

Figure 1.23

From the figure above, it is evident that $f(x) \to L$ as $x \to a$ from both sides. Thus, the limit $\lim_{x \to a} f(x) = L$ may informally be defined as "the limit of $f(x)$, as x approaches the value a, is the value L."

This definition, however, is vague. For example, what is meant by "$f(x)$ approaches the value L" or "x approaches the value a"? If this implies that the value of $f(x)$ gets closer and closer to L whenever x gets closer and closer to a, then how close is really "close"? How close is close enough to conclude that a limit is indeed approaching a certain value?

We deal with the idea of closeness by introducing the idea of error tolerance. That is, if we assign a value of error tolerance below which the distance between the function $f(x)$ and the value L must be kept, then we can keep x close

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enough to a in order for the distance between $f(x)$ and L to be less than that given error tolerance. The key idea is that the distance between $f(x)$ and L can be minimized to be smaller than any specified level of error tolerance by letting x sufficiently close to a.

We let ε be a small positive real number as a value of error tolerance. Since the error tolerance ε is the value less than which the distance between $f(x)$ and L must be, therefore the symbolic form of this is $|f(x) - L| < \varepsilon$. The absolute value is used to show that distance cannot be a negative quantity. Now, we derive the range within which $f(x)$ must lie as follows,

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ -\varepsilon < f(x) - L &< \varepsilon \\ -\varepsilon + L < f(x) - L + L &< \varepsilon + L \\ L - \varepsilon < f(x) &< L + \varepsilon \end{aligned}$$

This implies that $f(x)$ must lie inside an open interval $(L - \varepsilon, L + \varepsilon)$, as shown in the Figure 1.24.

Now, we let δ be the value less than which the distance between x and a must be kept in order for $|f(x) - L|$ to be less than the given error tolerance ε. The symbolic form of this is $|x - a| < \delta$. Based on that, we derive the range for x as follows,

$$\begin{aligned} |x - a| &< \delta \\ -\delta < x - a &< \delta \\ -\delta + a < x - a + a &< \delta + a \\ a - \delta < x &< a + \delta \end{aligned}$$

This implies that in order for the distance between $f(x)$ and L be less than a given error tolerance ε, x must be kept within an open interval $(a - \delta, a + \delta)$, where, $x \neq a$, as shown in the following figure.

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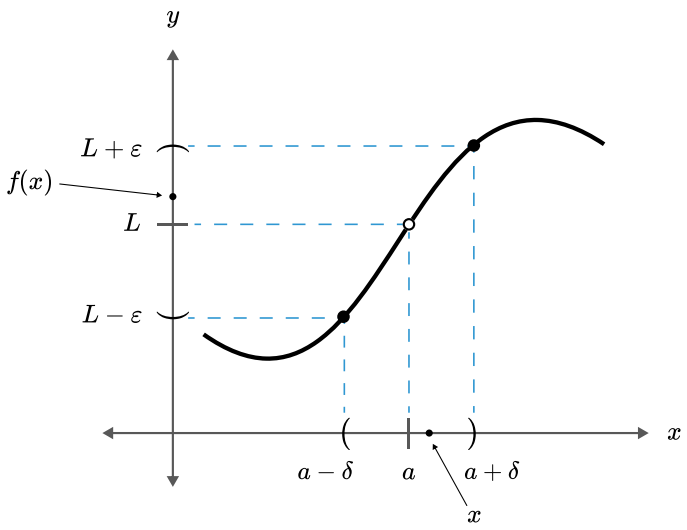

Figure 1.24

Since, $x \neq a$, therefore $|x - a| > 0$, and suggests the following,

$$\begin{aligned} \text{since,} \quad & 0 < |x - a| && \text{and,} \\ & |x - a| < \delta \\ \therefore \quad & 0 < |x - a| < \delta \end{aligned}$$

As a result, a rigorous definition of the limit of a function can be given as follows.

Definition 1.7.1. (The Formal Definition of a Limit)
Let $f(x)$ be a function defined over an open interval containing the point a (except possibly at a), and let L be a real number. Then the meaning of the limit

$$\lim_{x \to a} f(x) = L$$

is that for all every chosen $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon$$

A few examples are provided below.

► **Example 1.7.1.** Consider the function $f(x) = 2x - 5$. It is intuitively clear that $f(x) \to 1$ as $x \to 3$. That is,

$$\lim_{x \to 3} (2x - 5) = 1$$

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Now, what is the value less than which the distance between x and 3 must be kept in order for $|f(x) - 1|$ to be less than $\varepsilon = 0.0001$?

Solution. The value less than which the distance between x and 3 must be kept in order for $|f(x) - 1|$ to be less than 0.0001, is the value δ. That is, $|x - 3| < \delta$.

We know that, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Since, $f(x) = 2x - 5$ and $L = 1$, therefore,

$$\begin{aligned} |f(x) - L| < \varepsilon = |(2x - 5) - 1| &< 0.0001 \\ = |2x - 6| &< 0.0001 \\ = |2(x - 3)| &< 0.0001 \\ = 2|x - 3| &< 0.0001 \\ = |x - 3| &< \frac{0.0001}{2} \end{aligned}$$

This implies that $\delta = \dfrac{0.0001}{2} = 0.00005$ is the value less than which the distance between x and 3 must be kept in order for $|f(x) - 1|$ to be less than 0.0001. ∎

► **Example 1.7.2.** Verify the result derived for δ in Example 1.7.1.

Solution. In example 1.7.1, it was found that $\delta = 0.00005$. This implies that $|x - 3| < 0.00005$. Based on that, we derive,

$$\begin{aligned} |f(x) - L| &= |(2x - 5) - 1| \\ &= |2x - 6| \\ &= 2|x - 3| \\ &< 2(0.00005) = 0.0001 = \varepsilon \end{aligned}$$

This matches the error tolerance ε given in Example 1.7.1. This completes the verification. ∎

► **Example 1.7.3.** Prove that $\lim\limits_{x \to 2} (3x + 1) = 7$

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Solution. We can prove that $\lim_{x\to 2}(3x+1) = 7$, by showing that for any chosen $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|(3x+1)-7| < \varepsilon$.

Since $f(x) = 3x+1$ and $L = 7$, therefore,

$$\begin{aligned} |f(x) - L| < \varepsilon &= |(3x+1)-7| < \varepsilon \\ &= |3x-6| < \varepsilon \\ &= 3\,|x-2| < \varepsilon \\ &= |x-2| < \frac{\varepsilon}{3} \end{aligned}$$

This implies that if $0 < |x-2| < \dfrac{\varepsilon}{3}$, then $|(3x+1)-7| < \varepsilon$. Therefore, $\delta = \dfrac{\varepsilon}{3}$. ∎

► **Example 1.7.4.** Prove that $\lim_{x\to 1} x^2 = 1$

Solution. We can prove that $\lim_{x\to 1} x^2 = 1$, by showing that for any chosen $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|x^2 - 1| < \varepsilon$.

Since $f(x) = x^2$ and $L = 1$, therefore,

$$\begin{aligned} |f(x) - L| < \varepsilon &= \left|x^2 - 1\right| < \varepsilon \\ &= -\varepsilon < x^2 - 1 < \varepsilon \\ &= -\varepsilon + 1 < x^2 < \varepsilon + 1 \\ &= \sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon} \end{aligned}$$

Now, since $a = 1$, therefore,

$$\begin{aligned} |x-a| < \delta &= |x-1| < \delta \\ &= -\delta < x - 1 < \delta \\ &= 1 - \delta < x < 1 + \delta \end{aligned}$$

This implies that, either,

$$\begin{aligned} 1 - \delta &= \sqrt{1-\varepsilon} \\ \delta &= 1 - \sqrt{1-\varepsilon} \end{aligned}$$

or,

$$1 + \delta = \sqrt{1+\varepsilon}$$

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$$\delta = \sqrt{1+\varepsilon} - 1$$

Thus, we choose the smallest distance $\delta = \min\{1 - \sqrt{1-\varepsilon}, \sqrt{1+\varepsilon} - 1\}$.

This implies that if $0 < |x-1| < \min\{1 - \sqrt{1-\varepsilon}, \sqrt{1+\varepsilon} - 1\}$, then $|x^2 - 1| < \varepsilon$. ∎

► **Example 1.7.5.** Prove that $\lim_{x\to 4} \sqrt{x} = 2$

Solution. We can prove that $\lim_{x\to 4} \sqrt{x} = 2$, by showing that for any chosen $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-4| < \delta$, then $|\sqrt{x} - 2| < \varepsilon$.

Since $f(x) = \sqrt{x}$ and $L = 2$, therefore,

$$\begin{aligned} |f(x) - L| < \varepsilon &= \left|\sqrt{x} - 2\right| < \varepsilon \\ &= -\varepsilon < \sqrt{x} - 2 < \varepsilon \\ &= 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon \\ &= (2-\varepsilon)^2 < x < (2+\varepsilon)^2 \end{aligned}$$

Now, since $a = 4$, therefore,

$$\begin{aligned} |x - a| < \delta &= |x - 4| < \delta \\ &= -\delta < x - 4 < \delta \\ &= 4 - \delta < x < 4 + \delta \end{aligned}$$

This implies that, either,

$$\begin{aligned} 4 - \delta &= (2-\varepsilon)^2 \\ \delta &= 4 - (2-\varepsilon)^2 \end{aligned}$$

or,

$$\begin{aligned} 4 + \delta &= (2+\varepsilon)^2 \\ \delta &= (2+\varepsilon)^2 - 4 \end{aligned}$$

Thus, we choose the smallest distance $\delta = \min\{4 - (2-\varepsilon)^2, (2+\varepsilon)^2 - 4\}$.

This implies that if $0 < |x-4| < \min\{4-(2-\varepsilon)^2, (2+\varepsilon)^2 - 4\}$, then $|\sqrt{x} - 2| < \varepsilon$. ∎

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———————The End of Section I.7

$(\varepsilon - \delta)$ *Definition of a Limit*

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Part II

Differential Calculus

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Chapter 2

An Introduction to Differential Calculus

Differential calculus is a subfield of calculus that studies the rates at which quantities change at any given instant. In modern terms, it primarily addresses the question: how does a function react to a change in its input variable? For example, consider the function $f(x)$. We know that when the independent variable x changes, the dependent variable $f(x)$ also changes correspondingly, but the question that differential calculus essentially addresses is more about the "how" than the "how much." That is, it explores the nature of this change—to what sensitivity does $f(x)$ react to a change in x? What's the "rate[1]" at which $f(x)$ changes reacting to a change in x, no matter how small?

Finding the rate of change of $f(x)$ may seem straightforward: measure the amount x changes and compare it with the corresponding change in $f(x)$. For example, if x changed from the point $x = a$ to $x = a + h$, where $h > 0$ is an increment, then the change in x is the difference $(a + h) - a = h$ and the corresponding change in $f(x)$ for an increment h is the difference between $f(a+h)$ and $f(a)$, that is, $f(a + h) - f(a)$. Now, by comparing these two changes, we get,

$$\frac{f(a+h) - f(a)}{h} \tag{1}$$

[1] A rate is a specific type of ratio that compares two quantities of different units. It describes how one quantity changes relative to another.

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This quotient above represents the "rate" at which $f(x)$ changes as x changes from $x = a$ to $x = a+h$, and is called the *difference quotient* . So, for any change h in x in general, $f(x)$ changes at a rate of $\dfrac{f(x+h)-f(x)}{h}$. For example, if $f(x) = x$, then as x changes from $x = a$ to $x = a + h$, the change in $f(x)$ is,

$$\begin{aligned} f(a+h) - f(a) &= (a+h) - a \\ &= h \end{aligned}$$

Therefore, the rate at which $f(x)$ changes yields,

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{h}{h} \\ &= 1 \end{aligned}$$

This is because the function is $f(x) = x$, therefore the rate of change of $f(x)$ matches with the rate of change of x, resulting in a ratio of 1. It is also noticeable when the function is graphed, as shown below.

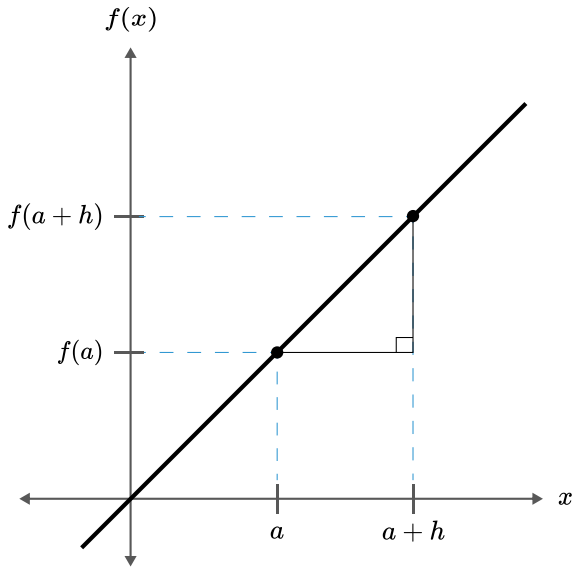

Figure 2.1

It is worth mentioning here is that, *when graphed*, the rate of change given by the difference quotient is essentially the same as the *slope* of the straight line. This is because we know that

$$\begin{aligned} \text{slope, } m &= \frac{\text{vertical change}}{\text{horizontal change}} \\ &= \frac{f(a+h)-f(a)}{(a+h)-a} \end{aligned}$$

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$$= \frac{f(a+h) - f(a)}{h}$$

Therefore, in general, the slope of a line on a graph represents the rate of change between two variables being graphed. In other words, it represents the rate at which the dependent variable changes reacting to a change in the independent variable.

With that in mind, let us now pose a "philosophical question" with the purpose to understand the tool through which differential calculus examines the rate of change of a function—namely, the derivative.

Philosophical Question: At what rate does $f(x)$ change *when there's no change in* x *whatsoever*?

This question is absurd! Why wouldn't it be? The dependent variable $f(x)$ only changes when the independent variable x changes. If the change in x is 0, then the change in $f(x)$ is also 0, and thus the rate at which $f(x)$ changes is... $\frac{0}{0}$? Can there be such a real[2] thing as $\frac{0}{0}$?

This absurd philosophical question is in fact equivalent to asking: what is the rate at which $f(x)$ changes at a point single point, say $x = a$. This question is equivalently absurd because if there's no change in x at the point $x = a$, then basically x changed from $x = a$ to $x = a + 0$, and the difference h between them is essentially 0, resulting again $\frac{0}{0}$. But why do we need to consider the rate of change of a function at a single point a?

This question can best be answered through problems that have perplexed solvers for centuries. One such problem is the *tangent line problem*. Understanding which can allow us to define the aforementioned heart of differential calculus: the derivative.

§ 2.1 The Tangent Line Problem

Given the graph of a function $f(x)$ and a point $P = (a, f(a))$ on the graph of $f(x)$ as shown below, can a line be drawn such that it is tangent to the graph of $f(x)$ at the point P?

[2] within the real number system.

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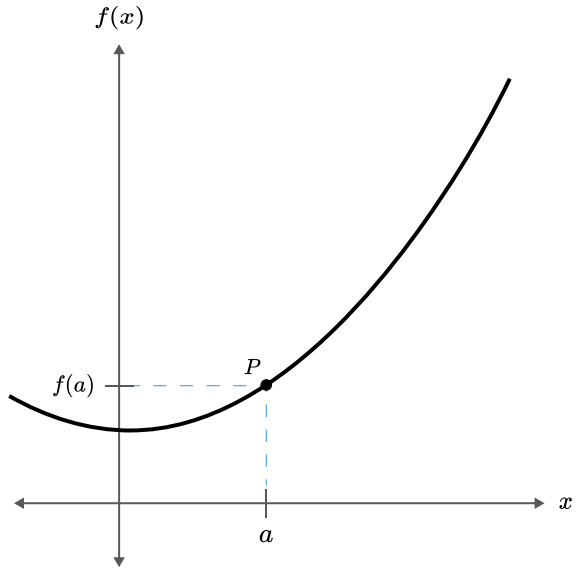

Figure 2.2

First of all, what does it mean for a line to be "tangent" to the graph of a function at a point? The term "tangent" originated from the Latin word "tangere," meaning "to touch"; so the term "tangent" is indicative of touching something. However, defining this precisely isn't as straightforward as it might appear. The definition of a tangent line is simple for common curves such as a circle. For example, a line is tangent to a circle if it meets the circle at exactly one and only point, known as the *point of tangency*, as shown below.

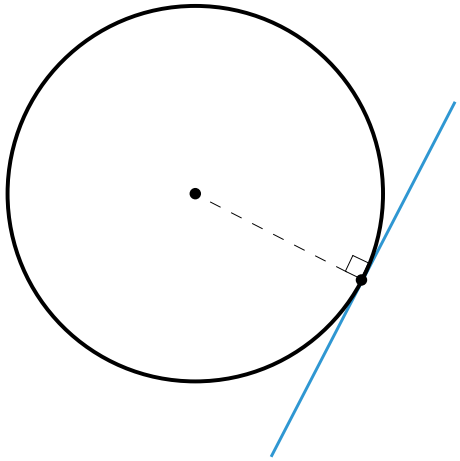

Figure 2.3

This intuitive definition might suggest that a line is tangent to a curve at a point if it "meets" the curve at that point. However, this definition only applies to simple curves like the one shown above and falls short for more complex curves. To understand why, consider the following figure.

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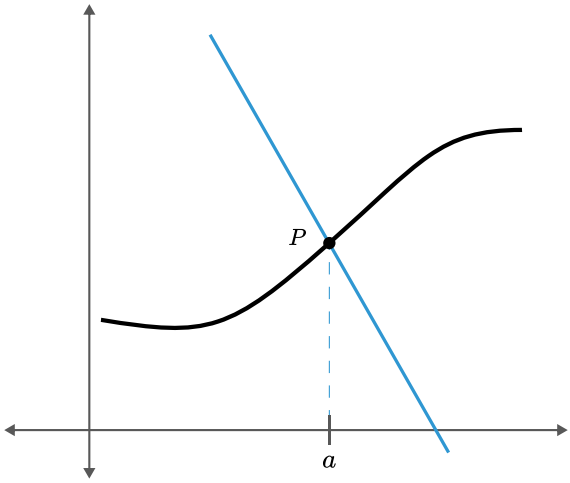

Figure 2.4

Here, the line "meets" the curve at exactly one point, but it goes without saying that it does not align with our previous understanding of a tangent line. Therefore, a revised definition would be that a line is tangent to a curve at a point if it "just touches" the curve at that point. This revised definition emphasizes the idea of a line gradually nearing the curve until it makes contact.

Sadly, however, this revised definition is also not sufficient for the reason depicted below.

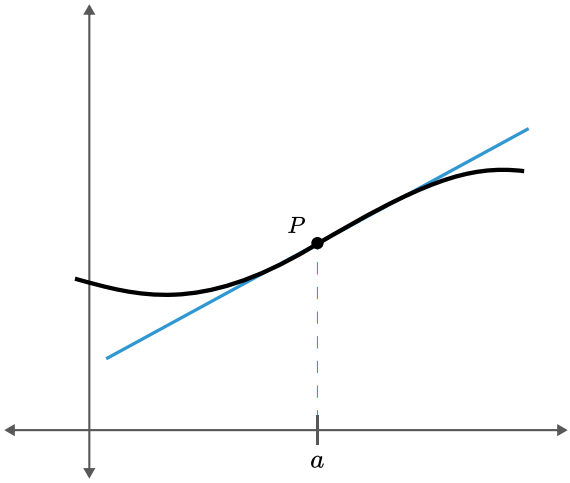

Figure 2.5

Here, the tangent line "just touches" the curve at only one point P, but it also crosses by intersecting the curve at that point. That is, a part of the curve lies above the tangent line while another part falls beneath it. Thus, it violates the current definition of a tangent line. Therefore, a revised definition would be that

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a line is tangent to a curve at a point if it either "just touches" or "intersects" the curve at exactly one point. However, even with this definition, the situation at hand is unsolved. Consider the figure below as an example,

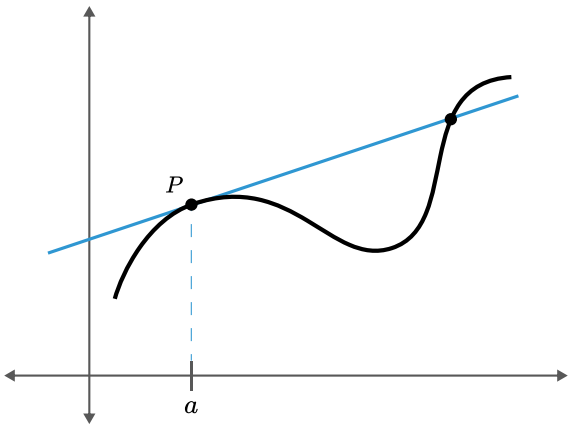

Figure 2.6

Here, the tangent line not only intersects the curve at point P, but also at another point ahead of P. This is again a violation of the current revised definition as it requires the point of intersection to be exactly one. Therefore, the definition can again be revised such that a line is tangent to a curve at a point if it, "up close," *just touches* or *intersects* the curve at that point. But unfortunately, even this definition eventually falls short.

This is frustrating. There's something inherently weird about defining a tangent line. This is because, in general, a tangent line requires no more than a single point for its definition (else it would become a secant line). However, to define a line, at least two points are required. Therefore, we're facing an seeming paradox: how can a tangent line be defined using only a single point?

That is, in order to define a line, at least two points are required. But if two points are used, then the line can no longer be defined as a tangent line—it becomes a secant line! A secant line to a curve is one that intersects the curve at two different points. It "cuts" the curve at least at two distinct points, as the term "secant" itself originated from the Latin word "secare," meaning "to cut."

It is at this point we may come to realize why the absurd philosophical question was posed earlier.

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To understand why, first recall that in Analytic Geometry a line can be defined using a single point if its slope is known. For example, recall the point-slope form of a line for a given point (x_1, y_1) and a slope m,

$$y - y_1 = m(x - x_1) \tag{1}$$

Now, if the slope m of the tangent line can somehow be determined, it then becomes possible to define the tangent line using only a single point by the equation (1). However, the core problem remains as the slope or the rate of change of two quantities at a single point becomes $\frac{0}{0}$, as previously explained.

Historically, efforts were made to calculate the slope of the tangent line to the curve at a single point by introducing a new quantity known as the "infinitesimal." To oversimplify, it was a quantity that was infinitely small but not exactly equal to 0. More specifically, it is a positive real number less than every positive real number. For example, consider the figure below,

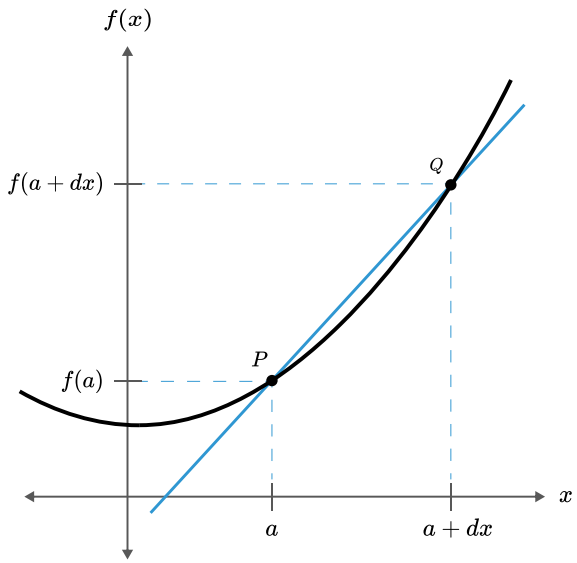

Figure 2.7

In order to calculate the slope of the tangent line to the curve of $f(x)$ at the point $P = (a, f(a))$, a small increment dx is made to $x = a$, resulting in a point $Q = (a + dx, f(a + dx))$ and then to draw a secant line PQ. Now, let dx be an infinitesimal increment. Since dx is an infinitesimal increment and infinitely small, therefore the point Q is infinitely closer to the point P in a way that they merge into one single point, becoming indistinguishable from each other, and thus, the slope of the secant line PQ is infinitely closer to the slope of the

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tangent line to the curve at the point P in a way that it is indistinguishably the same.

The unfortunate truth, however, is that the formulation above is—without a doubt—an invalid one within the framework of the real number system. There exists no such positive number that is less than every positive real number in the real number system. It can be proven. For example, if dx is a positive number smaller than any other positive number, where $dx > 0$. Then consider a real number $r > 1$ that divides dx, resulting $\frac{dx}{r} > 0$. Thus, we've found a new positive real number $\frac{dx}{r} > 0$ that contradicts the assumption that $dx > 0$ is the smallest positive real number because $dx > \frac{dx}{r} > 0$. This process can be repeated infinitely, disproving the existence of a positive number less than every positive real number.

In the historical development of calculus, mathematicians did indeed struggle with the concept of infinitesimals, leading some to criticize them by remarking, "Ghosts of departed Quantities." The accurate formulation of this concept was not possible until the rigorous formulation of the limit came into existence, namely the $(\varepsilon - \delta)$ definition of limits.

In modern calculus, we can simply calculate the slope of the tangent line by a limiting process where the point $Q = (a + dx, f(a + dx))$ approaches the point $P = (a, f(a))$ along the curve, i.e., the difference between $Q = (a+dx, f(a+dx))$ and $P = (a, f(a))$ approaches 0 as a function as $dx \to 0$, given that $dx \neq 0$.

More specifically, consider calculating the slope of the tangent line to the curve at point P, as shown in the figure below.

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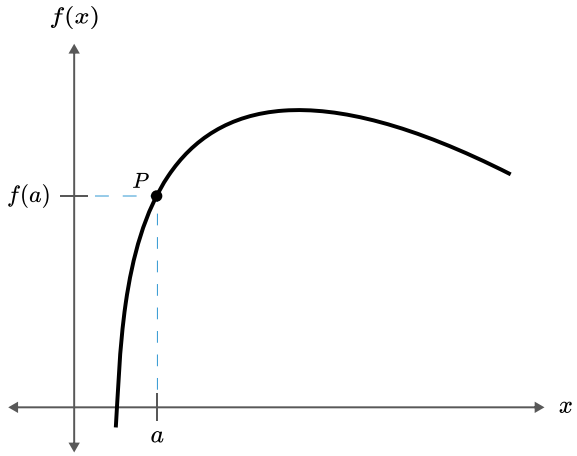

Figure 2.8

The slope can be determined by a limiting process using the slope of the secant line. To do this, let $Q = (a + h, f(a + h))$ be a second point on the curve that is close to $P = (a, f(a))$, where $h > 0$. Now, we draw the secant line PQ that intersects the points P and Q, as follows.

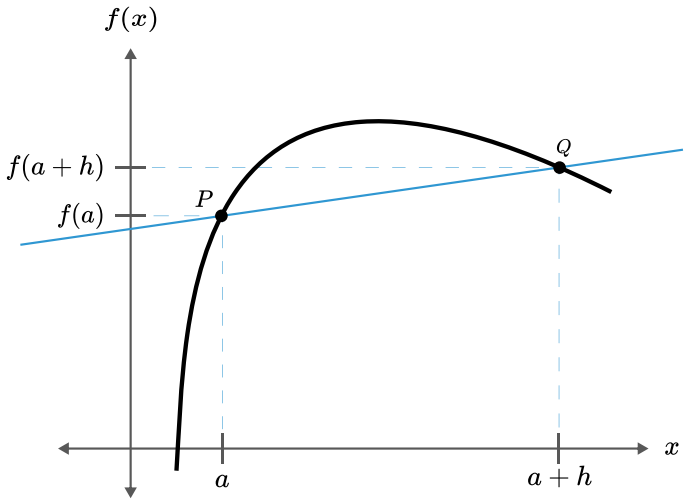

Figure 2.9

The slope of the line PQ is,

$$\begin{aligned} m_{\text{sec}} &= \frac{\text{vertical change}}{\text{horizontal change}} \\ &= \frac{f(a+h) - f(a)}{h} \end{aligned}$$

Since h is the difference between $x = a$ and $x = a+h$, therefore taking the limit[3]

[3]assuming the limit exists

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of the slope as $h \to 0$ causes the point Q to approach the point P by sliding along the curve of $f(x)$. The process is shown below.

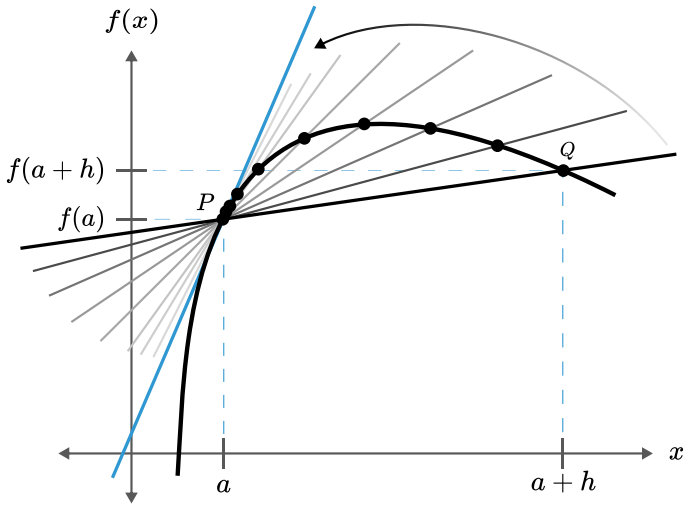

Figure 2.10

Thus, the slope of the tangent line is defined as

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \left(m_{\sec} \right) \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

In general, the slope of the tangent line to the graph of $f(x)$ at a point is defined using the slope of the secant line through a limiting process. Note that since the slope of a line is a measure of the rate of change between two points on that line, therefore the slope of the tangent line represents the "*instantaneous* rate of change." This is because the tangent line touches the curve at only one point, thus it represents the instantaneous rate of change at that precise instant.

Summarizing the facts discussed so far, we have the following definition,

Definition 2.1.1. (The Slope of a Tangent Line and The Instantaneous Rate of Change)

The slope of the curve $y = f(x)$ at a single point a equals the slope of the tangent line or the instantaneous rate of change of the curve at that point. Which is defined as,

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit exists, where $h > 0$ is a real number.

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Similarly, an alternative definition can be given when the point $a + h$ in Figure 2.10 is considered as a movable point x, as shown in the following figure.

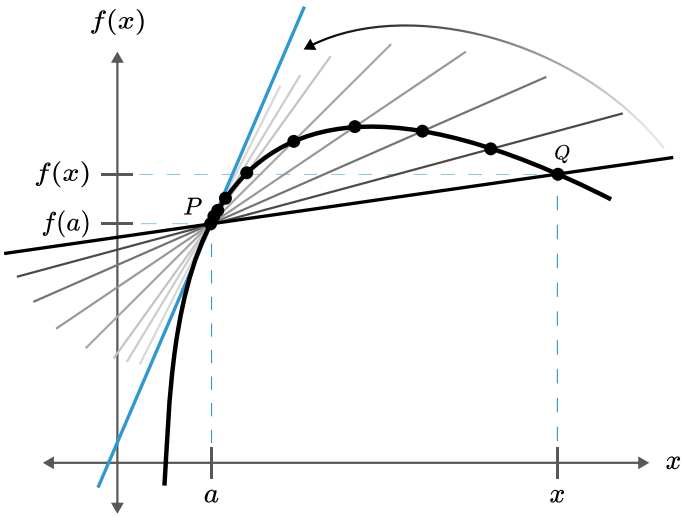

Figure 2.11

This time, Q approaches P if the moving point x approaches a. Thus, an alternative definition of the slope of the tangent line $m_{\tan}$ yields,

Definition 2.1.2. (The Slope of a Tangent Line and The Instantaneous Rate of Change)

The slope of the curve $y = f(x)$ at a single point a equals the slope of the tangent line or the instantaneous rate of change of the curve at that point. Which is defined as,

$$m_{\tan} = \lim_{x \to a} \frac{f(a) - f(x)}{x - a}$$

provided that the limit exists, where $x \neq a$.

Since we can now determine the slope of the tangent line $m_{\tan}$ at the point $P = (a, f(a))$ on the curve, we can define the tangent line using the point-slope form of a line as given by the equation 1 above. That is,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(a) &= m_{\tan}(x - a) \\ &= m_{\tan}(x - a) + f(a) \end{aligned}$$

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$$= \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) (x - a) + f(a)$$

Summarizing our discussion, we finally have the long-awaited definition of a line that is *tangent* to a curve at a single point as follows.

Definition 2.1.3. (Tangent Line) A line that is *tangent* to the graph of $f(x)$ at the point $P = (a, f(a))$ is given by,

$$y = m_{\tan}(x - a) + f(a)$$

where,

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit exists.

This is the generalized equation of a line that is tangent to the graph of $f(x)$ at the point a. Using this definition, the tangent line can be determined to the graph of any function at any point. For demonstration, consider the following example.

► **Example 2.1.1.** What is the tangent line to the graph of $f(x) = x^2$ at the point $(2, 4)$.

Solution. Since we've to find the tangent line at the point $(2, 4)$, therefore we first have to determine its slope at that point. From Definition 2.1.1, we know that the slope of the tangent line is given by,

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Therefore, the slope of the tangent line at the point $(2, 4)$ yields,

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} \\ &= \lim_{h \to 0} \frac{2^2 + 2 \cdot 2 \cdot h + h^2 - 2^2}{h} \\ &= \lim_{h \to 0} \frac{4h + h^2}{h} \\ &= \lim_{h \to 0} \frac{4h}{h} + \frac{h^2}{h} \end{aligned}$$

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$$\begin{aligned} &= \lim_{h\to 0} 4 + h \\ &= 4 \end{aligned}$$

Since $m_{\tan} = 4$, therefore the tangent line from Definition 2.1.3 yields,

$$\begin{aligned} y &= m_{\tan}(x - a) + f(a) \\ &= 4(x - 2) + 4 \\ &= 4x - 8 + 4 \\ &= 4x - 4 \end{aligned}$$

Therefore, $4x - 4$ is the line that is tangent to the graph of $f(x) = x^2$ at the point $(2, 4)$. ∎

Having discussed the tangent line problem and its resolution, we now move to discuss the derivative of a function in the following section and see how it is connected to the tangent line problem.

———————The End of Section 2.1
The Tangent Line Problem

§ 2.2 Definition of the Derivative of a function

We have, so far, calculated the slope of the tangent line to the curve at a fixed point. However, if that fixed point itself is not fixed and changes along the curve, then the tangent line also changes accordingly, resulting in changes to its slope for every new change. As a result, the slope of the tangent line to the curve of $f(x)$ becomes a new function on its own, dependent on the point to be evaluated. This new function has a special name called the "*derivative*" of $f(x)$. Notice how this new function is termed as a derivative "*of*" some other function. This is because the derivative is a function that is "derived" from some other function $f(x)$.

The derived function or the derivative can be expressed symbolically. For example, if $f(x)$ represents a function, then we let $f'(x)$ represent its derivative. Although many other notations exist and we will discuss them later on, but for now, we let $f'(x)$ (read as f prime of x) denote the derivative of a function $f(x)$.

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Since $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ represents the slope of the tangent line at a single point a, therefore,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

represents the slope of the tangent line at a *varying* point x, which is precisely said to be the derivative and thus denoted by $f'(x)$. Notice how the limit $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ can only be evaluated to find the slope of the tangent line when x is specified, and thus the derivative $f'(x)$ is indeed a function of x.

Summarizing the facts discussed, we have the following definition of the derivative of a function.

Definition 2.2.1. (The Derivative of a Function) The derivative $f'(x)$ of $f(x)$ at the point x is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided that the limit exists.

The derivative of a function can be thought of as a tool that helps us find the slope of the tangent line, i.e., the instantaneous rate of change of a function $f(x)$ at any given point. So, the derivative of a function may informally be said as follows.

"The derivative of a function is a mathematical tool used to measure the *rate* at which the function $f(x)$ changes with respect to a change in its input x *at any given instant*."

Think of it in this way: a curve does not maintain a constant slope at every point—it changes as you move along it. To determine the slope of the curve at a specific point, a tangent line is drawn to the curve at that point through a limiting process. The slope of this tangent line represents the curve's slope at that exact location. By calculating the slope at various points along the curve, using a variable x that moves across all points on the curve, we generate a new function called the "derivative." This derivative describes how the slope varies along the curve, representing the curve's instantaneous rate of change at each point.

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The definition of the derivative of a function can alternatively be defined as follows.

Definition 2.2.2. (The Derivative of a Function) The derivative of $f(x)$ at the point x is given by

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

To thoroughly understand the concept behind derivatives, let us "differentiate," that is, find the derivative of the function $f(x) = x^2$.

In order to do this, notice that

$$\begin{aligned} \text{for} \quad x \quad &\implies \quad f(x) = x^2 \\ \text{for} \quad x + h \quad &\implies \quad f(x+h) \quad = (x+h)^2 \\ &\qquad\qquad\quad = x^2 + 2xh + h^2 \end{aligned}$$

So, the change in y and x yields the following,

$$\begin{aligned} f(x+h) - f(x) = \underbrace{\left(x^2 + 2xh + h^2\right)}_{f(x+h)} - \underbrace{x^2}_{f(x)} \quad &\implies \quad 2xh + h^2 \\ (x+h) - x \quad &\implies \quad h \end{aligned}$$

Now, by comparing the changes as in the difference quotient, we get,

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2xh + h^2}{h} \\ &= \frac{2xh}{h} + \frac{h^2}{h} \\ &= 2x + h \end{aligned}$$

Since the derivative is defined as $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, therefore

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} (2x + h) \\ &= 2x + \lim_{h \to 0} h \\ &= 2x \end{aligned}$$

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This implies that the derivative of $f(x) = x^2$ is $f'(x) = 2x$—but what does it really mean? The function $f(x) = x^2$ defines a specific relationship between the input x and the output $f(x)$, resulting in a parabola visually. However, its derivative, $f'(x) = 2x$, explains how this relationship changes at every point along the curve. Specifically, it shows how the slope (i.e., the rate of change) of the parabola varies at different values of x. Basically, while $f(x) = x^2$ represents a parabolic curve that changes its slope continuously at different points, its derivative, $f'(x) = 2x$, tells us the exact rate of this change at those points. For example, at $x = 1$, the slope of $f(x) = x^2$ is $f'(1) = 2 \cdot 1 = 2$. At $x = 2$, the slope of $f(x) = x^2$ is $f'(2) = 2 \cdot 2 = 4$. At $x = 3$, the slope of $f(x) = x^2$ is $f'(2) = 2 \cdot 3 = 6$.

This is shown geometrically below.

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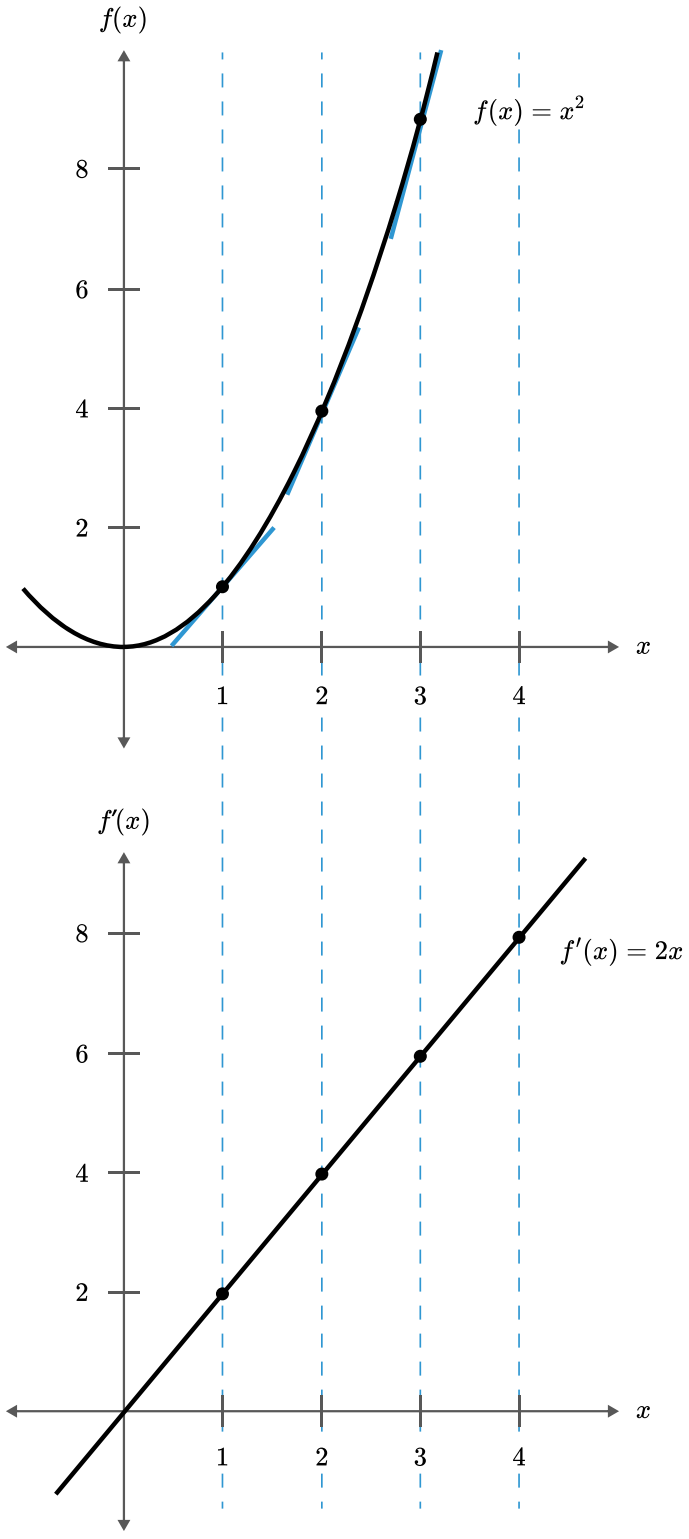

Figure 2.12

The figures above illustrate that the graph of $f(x) = x^2$ becomes progressively steeper as x increases. This suggests that the slope, or rate of change, also grows with every increase of x. As a result, the graph of the derivative $f'(x) = 2x$ of

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the function $f(x) = x^2$ also increases at a constant rate of 2 as x increases.

A few examples to derive the derivative of a function are provided below.

► **Example 2.2.1.** If $f(x) = 5x - 5$, then derive the derivative of $f(x)$.

Solution. By the definition of derivative, we get,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{[5(x+h) - 5] - (5x - 5)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5x} + 5h - \cancel{5} - \cancel{5x} + \cancel{5}}{h} \\
&= \lim_{h\to 0} \frac{5\cancel{h}}{\cancel{h}} \\
&= \lim_{h\to 0} 5 \\
&= 5
\end{aligned}$$

Therefore, the derivative of $f(x) = 5x - 5$ is $f'(x) = 5$. ∎

► **Example 2.2.2.** If $f(x) = \sqrt{x}$, then derive the derivative of $f(x)$.

Solution. By the definition of derivative, we get,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Therefore, the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$. ∎

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► **Example 2.2.3.** If $f(x) = x^3$, then derive the derivative of $f(x)$.

Solution. By the definition of derivative, we get,

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}\\
&= \lim_{h\to 0}\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}\\
&= \lim_{h\to 0}\frac{\cancel{h}\left(3x^2 + 3xh + h^2\right)}{\cancel{h}}\\
&= \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right)\\
&= \lim_{h\to 0} 3x^2 + \lim_{h\to 0} 3xh + \lim_{h\to 0} h^2\\
&= 3x^2 + 0 + 0\\
&= 3x^2
\end{aligned}$$

Therefore, the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$. ∎

► **Example 2.2.4.** If $f(x) = \dfrac{1}{x^2}$, then derive the derivative of $f(x)$.

Solution. By the definition of derivative, we get,

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{x^2(x+h)^2}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{-2xh - h^2}{x^2(x+h)^2}\right)\\
&= \lim_{h\to 0}\frac{1}{\cancel{h}}\cdot\frac{\cancel{h}(-2x - h)}{x^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{-2x - h}{x^2(x+h)^2}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{-2x - 0}{x^2(x+0)^2} \\
&= \frac{-2x}{x^2 \cdot x^2} \\
&= -\frac{2}{x^3}
\end{aligned}
$$

Therefore, the derivative of $f(x) = \frac{1}{x^2}$ is $f'(x) = -\frac{2}{x^3}$. ∎

———————The End of Section 2.2

Definition of the Derivative of a function

§ 2.3 Notation for Derivatives

In the previous section, the concept of the derivative of a function was introduced, and the notation $f'(x)$ was used to represent it. It is important to note, however, that several notations exist for the derivative, each rooted in its historical developments. These different notations are not merely a reflection of tradition but also offer practical advantages, as will become evident in their varied applications. One such convenient notation is the Leibniz notation , $\frac{dy}{dx}$, named after one of the co-founders of calculus: Gottfried Wilhelm Leibniz. To understand this, consider the following graph of a function that changed from x to $x + \Delta x$, resulting in a change from $f(x)$ to $f(x + \Delta x)$, where $\Delta x > 0$ is a small increment.

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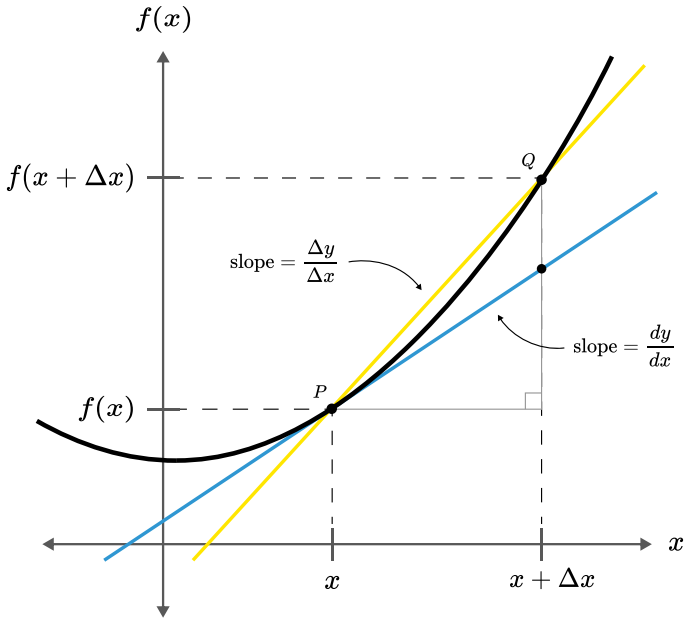

Figure 2.13

The slope of the secant line that passes through the points P and Q (the yellow colored line) is given by the difference quotient,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Since the change in y is often denoted as $\Delta y = f(x+\Delta x)-f(x)$, therefore, the slope of the secant line yields,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

Since the slope of the tangent line at a movable point x (i.e., the derivative) is defined as $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, therefore

$$\begin{aligned}\text{derivative, } f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \frac{dy}{dx}\end{aligned}$$

Here, $\dfrac{dy}{dx}$ is simply a single notation to represent the limit $\lim\limits_{\Delta\to 0}\dfrac{\Delta y}{\Delta x}$ is NOT a ratio of dy over dx. The notation $\dfrac{dy}{dx}$ is read as "the derivative of y with respect to x," or simply, "dee y dee x." Note that it is *never* read as "dee y divided by dee x" or "the ratio of dee y and dee x." The notation $\dfrac{dy}{dx}$ is not yet a quotient

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of two quantities dy and dx because the meaning of dy and dx have not yet been defined other than the fact that $\dfrac{dy}{dx}$ represents the limit $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. We will later consider giving dy and dx a meaning to set up a real quotient between them.

Two other equivalent forms of the Leibniz notation are as follows,

$$\frac{df(x)}{dx} \qquad \text{and} \qquad \frac{d}{dx}f(x)$$

The second notation "$\dfrac{d}{dx}(\quad)$" is thought of as an operator applied to a function $f(x)$ to perform differentiation, yielding its derivative $f'(x)$. This implies,

$$\frac{d}{dx}f(x) = f'(x)$$

In addition to these notations, there exist a few others as listed below.

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{df(x)}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

If it's shown that the derivative is evaluated at any point a, then the notations above are modified as follows.

$$f'(a) \qquad \left.\frac{dy}{dx}\right|_{x=a} \qquad \left.\frac{d}{dx}f(x)\right|_{x=a}$$

Considering all the different notations for derivatives at once may be overwhelming. However, it's important to note that in most cases, the Leibniz notation with its variants and the $f'(x)$ notation introduced earlier are the ones most commonly used. The notation $f'(x)$ is named after the Italian mathematician Joseph-Louis Lagrange.

———————The End of Section 2.3

Notation for Derivatives

§ 2.4 Differentiability and Continuity

In calculus, differentiability and continuity are two fundamental and related concepts, yet they are not interchangeable. If a function is continuous at a point then, informally, it does not have any sudden "jump" or "hole" at that point. On the other hand, differentiability indicates that the function has a well-defined tangent line (i.e., a derivative in broader terms) at that point.

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Not every continuous function has a derivative at every point. When a function $f(x)$ has a derivative $f'(x)$ at any given point x, the function is said to be *differentiable* at that point. Similarly, when $f(x)$ is differentiable at every point over an open interval I[4], it is said to be differentiable *over that interval* I.

Differentiability of a function at a point can be defined as follows.

Definition 2.4.1. (Differentiability) Let a be a point in the domain of the function $f(x)$. The function $f(x)$ is said to be differentiable at the point a if the following limit exists,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit, if it exists, is called the derivative of $f(x)$ at the point a.

A crucial result in the derivative of a function $f(x)$ lies in the fact that the differentiability of $f(x)$ implies its continuity. That is, if a function $f(x)$ is differentiable at the point $x = a$, i.e., if it has a derivative at the point $x = a$, then that function is necessarily continuous at that point $x = a$.

Theorem 2.4.1. (Differentiability Implies Continuity) If $f(x)$ is differentiable at a point a, then it must be continuous at that point.

Proof. Assume $f(x)$ is differentiable at a point $x = a$. If so, then the following limit exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Now, to show that $f(x)$ is not only differentiable but also continuous at $x = a$, we need to show the following,

$$\lim_{x \to a} f(x) = f(a)$$

In order to do this, consider the following.

$$\begin{aligned} f(x) &= f(x) \\ &= f(x) - \overbrace{f(a) + f(a)}^{0} \\ &= \Big(f(x) - f(a)\Big) \cdot \frac{x-a}{x-a} + f(a) \end{aligned}$$

[4]finite or infinite

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$$\therefore f(x) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)$$

Now, we take the limit as $x \to a$ on both sides of the equation above and simplify as follows.

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right] \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \underbrace{\lim_{x \to a} (x - a)}_{0} + \underbrace{\lim_{x \to a} f(a)}_{f(a)} \\
&= \underbrace{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}_{0} + f(a) \\
&= f(a) \\
\therefore \lim_{x \to a} f(x) &= f(a)
\end{aligned}$$

This implies that $f(x)$ is also continuous at the point $x = a$. This completes the proof. ∎

The theorem above states that if $f(x)$ is differentiable at a point $x = a$, then it necessarily is continuous at that point $x = a$. This logically follows that if $f(x)$ is not continuous at $x = a$, then it is necessarily not differentiable at $x = a$ and the following theorem states just that.

Theorem 2.4.2. (Not Continuous Implies Not Differentiable) If $f(x)$ is *not* continuous at a point a, then $f(x)$ is also *not* differentiable at that point.

Notice carefully that if $f(x)$ is continuous at a given point, then it may or may not be differentiable at that point. Conversely, if $f(x)$ is differentiable at a given point, then it is necessarily continuous at that point.

In practice, generally, a function $f(x)$ is not differentiable at a point a, if one of the following conditions occurs.

Remark 2.4.1.

1. $f(x)$ is not continuous at a.
2. $f(x)$ has a sharp corner at a.
3. $f(x)$ has a vertical tangent line at a.

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Consider the following example for a better insight.

► **Example 2.4.1.** Determine the differentiability and continuity of $f(x) = |x|$ at the point $x = 0$.

Solution. The function $f(x) = |x|$ is continuous at the point $x = 0$ because

$$\lim_{x\to 0} |x| = |0| = 0$$

The function, however, is not differentiable at $x = 0$, because the graph of $f(x)$ has a sharp cornet at $x = 0$. For this reason, the left-side and the right-side limits in the definition of the derivative of $f(x) = |x|$ are not equal. That is,

$$f'(0^-) = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$
$$f'(0^+) = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{+h}{h} = +1$$

Since $f'(0^-) \neq f'(0^+)$, therefore, the derivative does not exist at $x = 0$ as well. As a result, the function is continuous but not differentiable at $x = 0$. ∎

———————The End of Section 2.4
Differentiability and Continuity

§ 2.5 Higher-Order Derivatives

From the previous sections, we know that if $f(x)$ is a differentiable function, then $f'(x)$ represents its derivative. Notice that the derivative $f'(x)$ itself is a function that can be further differentiated if the derivative exists. Therefore, the derivative of the derivative of the function $f(x)$ is a second-order derivative of the function $f(x)$. It is denoted by $f''(x)$, as it can be thought of as $\Big(f'(x)\Big)' = f''(x)$. The process of taking derivatives can be continued endlessly, resulting in third, fourth, fifth, etc. order derivatives each time (provided that the function is differentiable each time). This notion can be expressed by the following notations.

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = \cdots = f^{(n)}(x)$$
$$y' = y'' = y''' = y^{(4)} = \cdots = y^{(n)}$$
$$\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = \frac{d^4y}{dx^4} = \cdots = \frac{d^ny}{dx^n}$$

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The second-order Leibniz notation $\dfrac{d^2y}{dx^2}$ can be viewed in terms of the first notation, that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

to mean that the operation of differentiation is applied to the first derivative of the function $y = f(x)$.

► **Example 2.5.1.** If $f(x) = x^3$ and $\dfrac{dy}{dx} = 3x^2$, then what is $\dfrac{d^2y}{dx^2}$?

Solution. The derivative of $\dfrac{dy}{dx} = 3x^2$ and the second-order derivative of $f(x) = x^3$ is,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} \\
&= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}\,(6x + 3h)}{\cancel{h}} \\
&= \lim_{h\to 0} (6x + 3h) \\
&= 6x + \lim_{h\to 0} 3h \\
&= 6x
\end{aligned}$$

This implies that if $f(x) = x^3$, then $\dfrac{d^2y}{dx^2} = 6x$. ∎

► **Example 2.5.2.** If $f(x) = x^3$, $\dfrac{dy}{dx} = 3x^2$, and $\dfrac{d^2y}{dx^2} = 6x$, then what is $\dfrac{d^3y}{dx^3}$?

Solution. The derivative of $\dfrac{d^2y}{dx^2} = 6x$ and the third-order derivative of $f(x) = x^3$ is,

$$\begin{aligned}
\frac{d^3y}{dx^3} &= \lim_{h\to 0} \frac{6(x+h) - 6x}{h} \\
&= \lim_{h\to 0} \frac{\cancel{6x} + 6h - \cancel{6x}}{h}
\end{aligned}$$

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$$\begin{aligned} &= \lim_{h\to 0} \frac{6\not{h}}{\not{h}} \\ &= \lim_{h\to 0} 6 \\ &= 6 \end{aligned}$$

This implies that if $f(x) = x^3$, then $\dfrac{d^3y}{dx^3} = 6$. ∎

► **Example 2.5.3.** If $f(x) = x^3$, $\dfrac{dy}{dx} = 3x^2$, $\dfrac{d^2y}{dx^2} = 6x$, and $\dfrac{d^3y}{dx^3} = 6$, then what is $\dfrac{d^4y}{dx^4}$?

Solution. The derivative of $\dfrac{d^3y}{dx^3} = 6$ and the fourth-order derivative of $f(x) = x^3$ is,

$$\begin{aligned} \frac{d^4y}{dx^4} &= \lim_{h\to 0} \frac{6-6}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

This implies that if $f(x) = x^3$, then $\dfrac{d^4y}{dx^4} = 0$. ∎

———————The End of Section 2.5

Higher-Order Derivatives

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Chapter 3

The Derivation of Derivatives

In the previous chapter, we discussed the concept of the derivative of a function and explored some examples to calculate them. Manually calculating these derivatives using the formal limit definition can be quite tedious. While the limit definition of derivative gives an exact method for computation, it is always better to have some shortcuts or general formulas to avoid reapplying the same limit process every time. These shortcuts are what we refer to as the differentiation rules.

The key idea is that once we derive the rules from the limit definition for common types of functions, we no longer need to perform complex limit calculations repeatedly, over and over again. Thus, we now begin to derive the differentiation rules for the most common types of functions in the following sections.

§ 3.1 Elementary Rules of Differentiation

In this section, we cover the fundamental differentiation rules that are widely used. These elementary rules allow us to calculate derivatives without requiring the limit definition and combine them to tackle more complex problems where limit-based methods would be difficult.

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Chap. 3 / Sec. 3.1 / Subsec. 3.1.1 : The Constant Rule

A function whose output is the same for all input is said to be a constant function. For example, the function $f(x) = c$, where c is any real number, is a constant function. Since the output of a constant function is always constant, therefore the graph of the constant function is a horizontal line across all input values, as shown below.

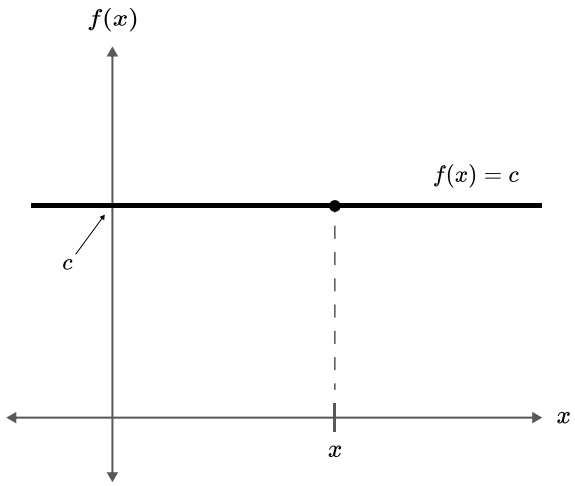

Figure 3.1

Since the slope of a horizontal line is 0, therefore the derivative of a constant function is simply 0. That is, if $f(x) = c$, where $c \in \mathbb{R}$, then $f'(x) = \dfrac{d}{dx}(c) = 0$. As a result, we have the following theorem.

Theorem 3.1.1. (The Constant Rule) If $c \in \mathbb{R}$, then

$$\frac{d}{dx}(c) = 0$$

Proof. If $f(x) = c$, where $c \in \mathbb{R}$, then $f(x + h) = c$. Now, by the definition of the derivative of a function, we write,

$$\begin{aligned}
\frac{d}{dx}(c) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0
\end{aligned}$$

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This completes the proof. ∎

A few examples of the theorem above are presented below.

► **Example 3.1.1.** If $f(x) = 27$, then what is the derivative of $f(x)$?

Solution. The derivative of $f(x) = 27$ is as follows,

$$\begin{aligned} f'(x) &= \frac{d}{dx} f(x) \\ &= \frac{d}{dx}(27) \\ &= 0 \end{aligned}$$

$\therefore$ If $f(x) = 27$, then $f'(x) = 0$. ∎

► **Example 3.1.2.** If $f(x) = \pi$, then what is $f'(x)$?

Solution. Note that π is a constant, therefore the derivative of $f(x) = \pi$ is as follows,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\pi) \\ &= 0 \end{aligned}$$

$\therefore$ If $f(x) = \pi$, then $f'(x) = 0$. ∎

► **Example 3.1.3.** If $f(x) = \pi^e$, then $f'(x) =?$

Solution. The derivative of $f(x) = \pi^e$ is as follows,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\pi^e\right) \\ &= 0 \end{aligned}$$

$\therefore$ If $f(x) = \pi^e$, then $f'(x) = 0$. ∎

Chap. 3 / Sec. 3.1 / Subsec. 3.1.2 : The Constant Multiple Rule

This rule is obvious. The derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of that function. That is,

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Theorem 3.1.2. (The Constant Multiple Rule) If $c \in \mathbb{R}$ and $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\big(c \cdot f(x)\big) = c \cdot \frac{d}{dx} f(x)$$

Proof. We will prove this directly.

$$\begin{aligned}
\frac{d}{dx}\left(c \cdot f(x)\right) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\
&= \lim_{h \to 0} \left[c \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right] \\
&= \lim_{h \to 0} (c) \cdot \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\
&= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= c \cdot f'(x)
\end{aligned}$$

This completes the proof. ∎

A few examples of the theorem above are presented below.

► **Example 3.1.4.** If $f(x) = x^2$, then what is the derivative of $\dfrac{x^2}{e}$, where e is a constant and $\dfrac{d}{dx}\left(x^2\right) = 2x$?

Solution. Applying Theorem 3.1.2, we get,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2}{e}\right) &= \frac{d}{dx}\left(\frac{1}{e} \cdot x^2\right) \\
&= \frac{1}{e} \cdot \frac{d}{dx}\left(x^2\right) \\
&= \frac{1}{e} \cdot 2x \\
&= \frac{2x}{e}
\end{aligned}$$

Therefore, the derivative of the function $f(x) = \dfrac{x^2}{e}$ is $f'(x) = \dfrac{2x}{e}$. ∎

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Chap. 3 / Sec. 3.1 / Subsec. 3.1.3 : The Power Rule

Functions of the form $f(x) = x^n$, where $n \in \mathbb{N}$, are called the power functions. Notice that if $n = 1$, then $f(x) = x^n$ simplifies to $f(x) = x^1 = x$, representing a straight line with a slope of 1. Therefore, the derivative of the function $f(x) = x$ is 1, as shown below.

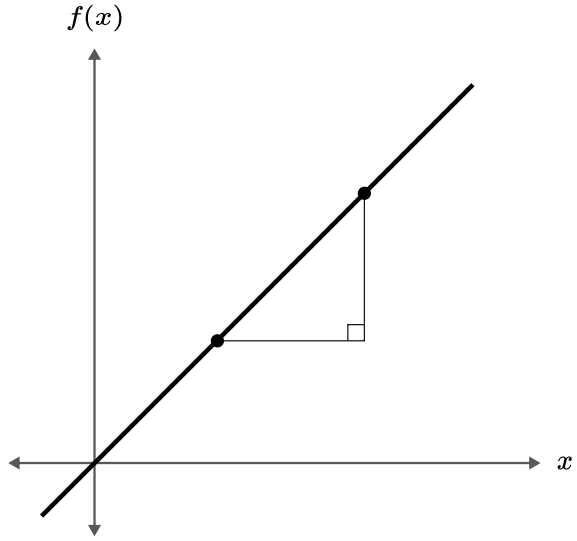

Figure 3.2

Theorem 3.1.3. (Derivative of Identity Function) If $f(x) = x$ is a differentiable function, then

$$\frac{d}{dx}(x) = 1$$

However, for $n > 1$, there isn't a specific general case such as the one mentioned above. For example, how are the derivatives calculated for functions such as $f(x) = x^2$ or $f(x) = x^3$? Currently, we can only apply the definition of derivatives to these functions to determine their derivatives. For example, to determine the derivative of $f(x) = x^2$, we proceed with the following approach.

If $f(x) = x^2$, then $f(x + h) = (x + h)^2$ and it is known that,

$$(a + b)^2 = a^2 + 2ab + b^2$$

Therefore, we apply the definition of derivative to determine the derivative of $f(x) = x^2$ as follows,

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

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$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x+h) \\
&= 2x
\end{aligned}
$$

Now, if $f(x) = x^3$, then $f(x+h) = (x+h)^3$ and it is known that,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Therefore, we similarly apply the definition of a derivative to determine the derivative of $f(x) = x^3$ as follows,

$$
\begin{aligned}
\frac{d}{dx} f(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}
$$

A pattern may be visible. That is, if $f(x) = x^2$, then $f'(x) = 2x$. Similarly, if $f(x) = x^3$, then $f'(x) = 3x^2$. Indeed, this pattern suggests that if we have a function $f(x)$ defined as x raised to the power of n, where n is a positive integer, then its derivative equals n multiplied by x raised to the power of $n-1$. This observation can be proven.

Theorem 3.1.4. (The Power Rule) If $f(x) = x^n$ is a differentiable function, where $n \in \mathbb{N}$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Before presenting the proof of the theorem above, we first need to review the Binomial Theorem.

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We know the following,

$$\begin{aligned}(x+h)^2 &= x^2 + 2xh + h^2\\ (x+h)^3 &= x^3 + 3x^3h + 3xh^2 + h^3\\ (x+h)^4 &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4\\ (x+h)^5 &= x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5\end{aligned}$$

Likewise, the general expansion of a binomial expression raised to nth power is the following,

$$(x+h)^n = x^n + nx^{n-1}\cdot(h) + \frac{n(n-1)x^{n-2}}{2}\cdot(h)^2 + \cdots + (h)^n$$

Having this at hand, we now begin to prove Theorem 3.1.4 as follows,

Proof. We will prove this theorem directly.

$$\begin{aligned}\frac{d}{dx}\left(x^n\right) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{\left(x^n + nx^{n-1}\cdot(h) + \frac{n(n-1)}{2}(x^{n-2})\cdot(h)^2 + \cdots + (h)^n\right) - x^n}{h}\\ &= \lim_{h\to 0}\frac{\cancel{x^n} + \cancel{h}\left(nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}\cdot(h) + \cdots + (h)^{n-1}\right) - \cancel{x^n}}{\cancel{h}}\\ &= \lim_{h\to 0}\ nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}\cdot h + \cdots + h^{n-1}\\ &= nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}\cdot 0 + \cdots + 0^{n-1}\\ &= nx^{n-1} + 0 + \cdots + 0\\ &= nx^{n-1}\end{aligned}$$

This completes the proof. ∎

A few examples of the theorem above are presented below.

► **Example 3.1.5.** If $f(x) = x^{999}$, then $f'(x) = ?$

Solution. Applying the Power Rule, we get,

$$\frac{d}{dx}\left(x^{999}\right) = 999x^{999-1}$$

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$$= 999x^{998}$$

Therefore, $f'(x) = 999x^{998}$. ∎

► **Example 3.1.6.** If $f(x) = x^{1+2+3+4+5+6+7+8+9}$, then what is $\dfrac{dy}{dx}$?

Solution. Since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, therefore, applying the Power Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^{45}\right)\\ &= 45x^{45-1}\\ &= 45x^{44}\end{aligned}$$

Therefore, if $f(x) = x^{1+2+3+4+5+6+7+8+9}$, $\dfrac{dy}{dx} = 45x^{44}$. ∎

Chap. 3 / Sec. 3.1 / Subsec. 3.1.4 : The Summation Rule

When examining a variety of functions, it's often the case that they are formulated as combinations of simpler functions. Therefore, it is important for us to have a systematic method for finding the derivative of these combined functions. One such combined function is obtained through the mathematical operation of addition. That is, if $f(x)$ and $g(x)$ are two functions that can be easily differentiated, then differentiation of their sum $f(x) + g(x)$ follows the theorem below.

Theorem 3.1.5. (The Summation Rule) If $f(x)$ and $g(x)$ are both differentiable functions, then

$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof. We will prove this directly.

$$\begin{aligned}\frac{d}{dx}\Big(f(x) + g(x)\Big) &= \lim_{h\to 0}\frac{\Big(f(x+h) + g(x+h)\Big) - \Big(f(x) + g(x)\Big)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}\end{aligned}$$

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$$
\begin{aligned}
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)\\
&= \frac{d}{dx}f(x)+\frac{d}{dx}g(x)
\end{aligned}
$$

This completes the proof. ∎

This summation rule can be extended for more than two differentiable functions such as $f_1(x), f_2(x), f_3(x), \cdots, f_{n-1}, f_n(x)$, yielding a generalized summation rule as follows.

$$\frac{d}{dx}\Big(f_1(x)+f_2(x)+\cdots+f_n(x)\Big)=\frac{d}{dx}f_1(x)+\frac{d}{dx}f_2(x)+\cdots+\frac{d}{dx}f_n(x)$$

A few examples of the theorem above are presented below.

► **Example 3.1.7.** What is the derivative of $f(x)=2x^2+3x^3+5x^5+7x^7$?

Solution. Since the derivatives of $2x^2$, $3x^3$, $5x^5$, $7x^7$ are $4x$, $9x^2$, $25x^4$, $49x^6$, respectively, by the Power Rule. Therefore, the derivative of $f(x)=2x^2+3x^3+5x^5+7x^7$ yields,

$$
\begin{aligned}
\frac{d}{dx}\left(2x^2+3x^3+5x^5+7x^7\right) &= \frac{d}{dx}2x^2+\frac{d}{dx}3x^3+\frac{d}{dx}5x^5+\frac{d}{dx}7x^7\\
&= \frac{d}{dx}2x^2+\frac{d}{dx}3x^3+\frac{d}{dx}5x^5+\frac{d}{dx}7x^7\\
&= 4x+9x^2+25x^4+49x^6
\end{aligned}
$$

Therefore, $\dfrac{d}{dx}\left(2x^2+3x^3+5x^5+7x^7\right)=4x+9x^2+25x^4+49x^6$ ∎

► **Example 3.1.8.** If $\dfrac{d}{dx}\Big(x^2\Big)=2x$ and $\dfrac{d}{dx}\Big(x^3\Big)=3x^2$, then what is $\dfrac{d}{dx}x(x+x^2)$?

Solution. First, notice that, since $x(x+x^2)=x^2+x^3$, therefore $\dfrac{d}{dx}x(x+x^2)=\dfrac{d}{dx}\Big(x^2+x^3\Big)$. Thus,

$$\frac{d}{dx}\Big(x^2+x^3\Big)=\frac{d}{dx}\Big(x^2\Big)+\frac{d}{dx}\Big(x^3\Big)$$

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$$= 2x + 3x^2$$
$$= x(2 + 3x)$$

Therefore, the derivative of $f(x) = x\big(x + x^2\big)$ is $f'(x) = x(3x + 2)$. ∎

Chap. 3 / Sec. 3.1 / Subsec. 3.1.5 : The Difference Rule

Similar to the Summation Rule, the Difference Rule is also used to find the derivative of combined functions that involve the mathematical operation of subtraction. If $f(x)$ and $g(x)$ are differentiable functions, then differentiation of $f(x) - g(x)$ follows the theorem below.

Theorem 3.1.6. (The Difference Rule) If $f(x)$ and $g(x)$ are both differentiable functions, then

$$\frac{d}{dx}\Big(f(x) - g(x)\Big) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof. We shall prove this directly as follows.

$$\begin{aligned}
\frac{d}{dx}\Big(f(x) - g(x)\Big) &= \lim_{h\to 0} \frac{\Big(f(x+h) - g(x+h)\Big) - \Big(f(x) - g(x)\Big)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) - g'(x) \\
&= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)
\end{aligned}$$

This completes the proof. ∎

The difference rule can also be extended for more than two differentiable functions such as $f_1(x), f_2(x), f_3(x), \cdots, f_{n-1}, f_n(x)$, yielding a generalized difference rule as follows.

$$\frac{d}{dx}\Big(f_1(x) - f_2(x) - \cdots - f_n(x)\Big) = \frac{d}{dx}f_1(x) - \frac{d}{dx}f_2(x) - \cdots - \frac{d}{dx}f_n(x)$$

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A few examples of the theorem above are presented below.

► **Example 3.1.9.** What is the derivative of $f(x) = x^1 - x^2 - x^3 - x^4 - x^5$?

Solution. Since the derivatives of x^1, x^2, x^3, x^4, and x^5 are 1, $2x$, $3x^2$, $4x^3$, and $5x^4$, respectively, by the Power Rule. Therefore, the derivative of $f(x) = x^1 - x^2 - x^3 - x^4 - x^5$ yields,

$$\begin{aligned}\frac{d}{dx}\left(x^1 - x^2 - x^3 - x^4 - x^5\right) &= \frac{d}{dx}x^1 - \frac{d}{dx}x^2 - \frac{d}{dx}x^3 - \frac{d}{dx}x^4 - \frac{d}{dx}x^5\\ &= 1 - 2x^1 - 3x^2 - 4x^3 - 5x^4\end{aligned}$$

Therefore, $\dfrac{d}{dx}\left(x^1 - x^2 - x^3 - x^4 - x^5\right) = 1 - 2x^1 - 3x^2 - 4x^3 - 5x^4$ ∎

► **Example 3.1.10.** What is the derivative of

$$f(x) = ex^1 - \pi xx^{1+1} - exxx^{1+1+1} - \pi xxxx^{1+1+1+1} - exxxxx^{1+1+1+1+1}?$$

Solution. The function $f(x) = ex^1 - \pi xx^{1+1} - exxx^{1+1+1} - \pi xxxx^{1+1+1+1} - exxxxx^{1+1+1+1+1}$ can be simplified as

$$f(x) = ex^1 - \pi x^3 - ex^5 - \pi x^7 - ex^9$$

where, e and π are constant. Now, by the Difference Rule, Constant Multiple Rule, and Power Rule; we get,

$$\begin{aligned}\frac{d}{dx}\left(ex^1 - \pi x^3 - ex^5 - \pi x^7 - ex^9\right) &= \frac{d}{dx}ex^1 - \frac{d}{dx}\pi x^3 - \frac{d}{dx}ex^5 - \frac{d}{dx}\pi x^7 - \frac{d}{dx}ex^9\\ &= e\frac{d}{dx}x^1 - \pi\frac{d}{dx}x^3 - e\frac{d}{dx}x^5 - \pi\frac{d}{dx}x^7 - e\frac{d}{dx}x^9\\ &= e - 3\pi x^2 - 5ex^4 - 7\pi x^6 - 9ex^8\end{aligned}$$

Therefore, the derivative of $f(x) = ex^1 - \pi xx^{1+1} - exxx^{1+1+1} - \pi xxxx^{1+1+1+1} - exxxxx^{1+1+1+1+1}$ is $f'(x) = e - 3\pi x^2 - 5ex^4 - 7\pi x^6 - 9ex^8$. ∎

Chap. 3 / Sec. 3.1 / Subsec. 3.1.6 : The Product Rule

In the previous combined functions, we've seen that the derivative of a sum of functions equals the sum of their derivatives, and the derivative of a difference

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of functions equals the difference of their derivatives. This might lead to an assumption that the derivative of a product of functions equals the product of their derivatives, but this is not the case this time. For example, if $f(x) = x^2$ and $g(x) = x^3$, then the combined function yields $H(x) = f(x) \cdot g(x) = x^2 \cdot x^3 = x^5$. From the Power Rule, we know that the derivative of $H(x) = x^5$ is $H'(x) = 5x^4$. However, if the derivative of a product of functions were to be the product of their derivatives, then the derivative of the combined function $H(x) = x^2 \cdot x^3$ would've been $H'(x) = 2x \cdot 3x^2 = 6x^3$, which is invalid.

To understand the logic behind it, we can take a visual approach to explore the underlying factors as follows.

Assume that $f(x)$ and $g(x)$ are both differentiable functions. Then the product of these functions, $f(x) \cdot g(x)$ can be interpreted as an area of a rectangle as shown below.

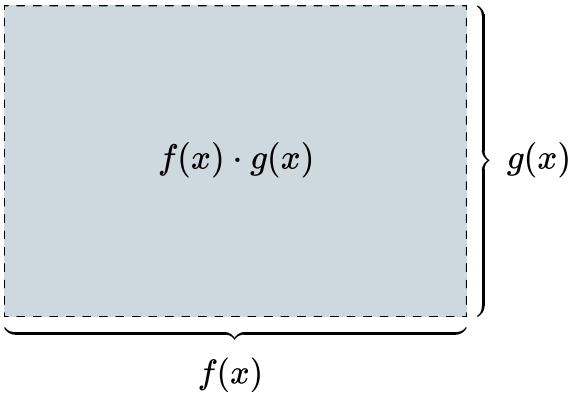

Figure 3.3

Now, if x changes by an increment of Δx, then the corresponding changes in $f(x)$ and $g(x)$ are as follows.

$$\Delta f = f(x + \Delta x) - f(x) \qquad \text{and} \qquad \Delta g = g(x + \Delta x) - g(x)$$

This is shown in the following figure.

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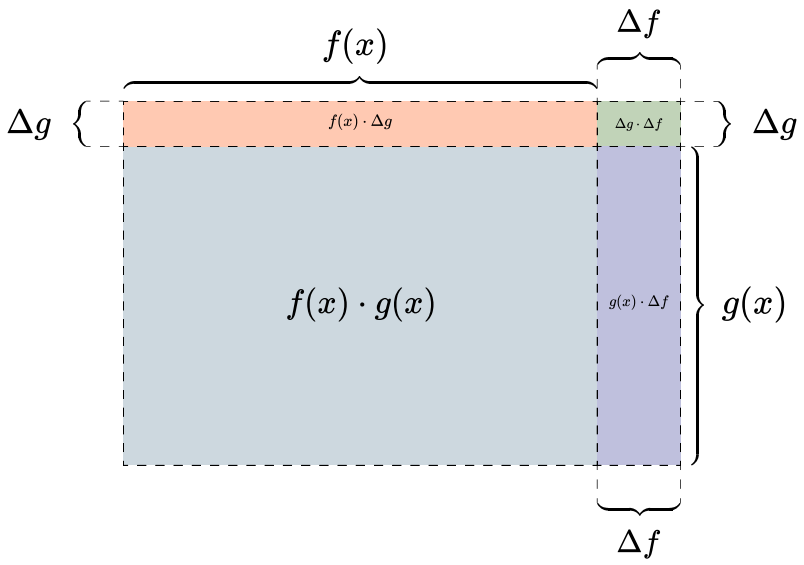

Figure 3.4

Now, the product $\Big(f(x)+\Delta f\Big)\cdot\Big(g(x)+\Delta g\Big)$ represents the area of the large rectangle. And the change in the area $f(x)\cdot g(x)$ of the initial rectangle, that is,

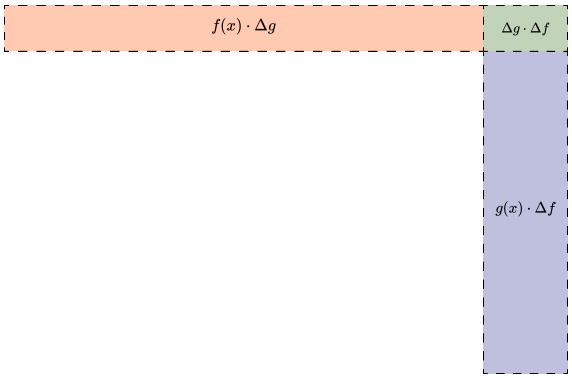

Figure 3.5

is the following,

$$\begin{aligned}
\Delta\Big(f(x)\cdot g(x)\Big) &= \Big(f(x)+\Delta f\Big)\cdot\Big(g(x)+\Delta g\Big)-\Big(f(x)\cdot g(x)\Big)\\
&= \Big(f(x)\cdot g(x)+f(x)\cdot\Delta g+\Delta f\cdot g(x)+\Delta f\cdot\Delta g\Big)-\Big(f(x)\cdot g(x)\Big)\\
&= \cancel{f(x)\cdot g(x)}+f(x)\cdot\Delta g+\Delta f\cdot g(x)+\Delta f\cdot\Delta g-\cancel{f(x)\cdot g(x)}\\
&= f(x)\cdot\Delta g+\Delta f\cdot g(x)+\Delta f\cdot\Delta g \qquad (1)
\end{aligned}$$

Now to get the rate of the change, we compare the quantities by dividing equation (1) with Δx, that is,

$$\frac{\Delta\Big(f(x)\cdot g(x)\Big)}{\Delta x}=\frac{f(x)\cdot\Delta g+\Delta f\cdot g(x)+\Delta f\cdot\Delta g}{\Delta x}$$

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$$= \frac{f(x) \cdot \Delta g}{\Delta x} + \frac{\Delta f \cdot g(x)}{\Delta x} + \frac{\Delta f \cdot \Delta g}{\Delta x}$$
$$= f(x) \cdot \frac{\Delta g}{\Delta x} + g(x) \cdot \frac{\Delta f}{\Delta x} + \Delta f \cdot \frac{\Delta g}{\Delta x}$$

Now, since the derivative is defined as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, there we write the following to get the instantaneous rate of change (i.e., the derivative) of the product $f(x) \cdot g(x)$,

$$= \frac{d}{dx}\left(f(x) \cdot g(x)\right)$$
$$= \lim_{\Delta x \to 0} \frac{\Delta\Big(f(x) \cdot g(x)\Big)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(f(x) \cdot \frac{\Delta g}{\Delta x} + g(x) \cdot \frac{\Delta f}{\Delta x} + \Delta f \cdot \frac{\Delta g}{\Delta x}\right)$$
$$= f(x) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}\right) + g(x) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \Delta f\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}\right)$$
$$= f(x) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}\right) + g(x) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} f\left(x + \Delta x\right) - f\left(x\right)\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}\right)$$
$$= f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x) + 0 \cdot \frac{d}{dx} g(x)$$
$$= f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

This implies that the derivative of the product of two functions $f(x)$ and $g(x)$ is not merely the product of their derivatives, that is, $\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) \neq \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$, but rather, $\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$.

Therefore, we have the following theorem.

Theorem 3.1.7. (The Product Rule) If $f(x)$ and $g(x)$ are both differentiable functions, then

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

Proof. We will prove this directly,

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big)$$
$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

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$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) \overbrace{-f(x+h)\cdot g(x) + f(x+h)\cdot g(x)}^{0} - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x)}{h} + \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h} \right) \\
&= \lim_{h\to 0} \left[f(x+h)\cdot \left(\frac{g(x+h)-g(x)}{h} \right) + g(x)\cdot \left(\frac{f(x+h)-f(x)}{h} \right) \right] \\
&= \lim_{h\to 0} \left(f(x+h)\cdot \frac{g(x+h)-g(x)}{h} \right) + \lim_{h\to 0} \left(g(x)\cdot \frac{f(x+h)-f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= f(x)\cdot g'(x) + g(x)\cdot f'(x)
\end{aligned}
$$

This completes the proof. ∎

Compared to the theorems previously discussed, this theorem seems to be quite convoluted. Therefore, it is often helpful to simplify it in the following equivalent manner.

Remark 3.1.1. If $f(x) = u$ and $g(x) = v$ are both differentiable functions, then

$$\frac{d}{dx}(u\cdot v) = u\cdot \frac{d}{dx}(v) + v\cdot \frac{d}{dx}(u)$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.1.11.** What is the derivative of $f(x) = ex(x+1)$?

Solution. If we let $u = ex$ and $v = x+1$, then by the Product Rule,

$$
\begin{aligned}
\frac{d}{dx} ex(x+1) &= ex\cdot \frac{d}{dx}(x+1) + (x+1)\cdot \frac{d}{dx} ex \\
&= ex + (x+1)\cdot e \\
&= ex + ex + e \\
&= 2ex + e \\
&= e(2x+1)
\end{aligned}
$$

Therefore, the derivative of $f(x) = ex(x+1)$ is $f'(x) = e(2x+1)$. ∎

► **Example 3.1.12.** Verify $\frac{d}{dx} ex(x+1) = e(2x+1)$ in the previous example without using the Product Rule.

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Solution. Notice that $ex(x+1) = ex^2 + ex$, therefore, by the Power Rule, we get,

$$\begin{aligned}\frac{d}{dx}\left(ex^2 + ex\right) &= \frac{d}{dx}ex^2 + \frac{d}{dx}ex \\ &= e\frac{d}{dx}x^2 + e\frac{d}{dx}x \\ &= 2xe + e \\ &= e(2x+1)\end{aligned}$$

This confirms that $\frac{d}{dx}ex(x+1) = e(2x+1)$ ∎

Chap. 3 / Sec. 3.1 / Subsec. 3.1.7 : The Quotient Rule

Similar to the Product Rule, the derivative of a quotient of functions does not equal the quotient of their derivatives. It is also clear to see, for example, if $f(x) = x^3$ and $g(x) = x^2$, then the quotient yields $H(x) = \frac{f(x)}{g(x)} = \frac{x^3}{x^2} = x$. Therefore, the derivative of $H(x)$ yields, $H'(x) = 1$. Which is not equal to the quotient of the function's derivatives, that is, $H'(x) \neq \frac{3x^2}{2x} = \frac{3x}{2}$.

To differentiate a quotient of functions, the Quotient Rule can be derived from the Product Rule. For example, consider the quotient function $q(x) = \frac{f(x)}{g(x)}$. This implies that $f(x) = g(x) \cdot q(x)$. From the Product Rule, we know that,

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}g(x) \cdot q(x) \\ &= q(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}q(x)\end{aligned}$$

Solving for $\frac{d}{dx}q(x)$ yields,

$$\frac{d}{dx}q(x) = \frac{\frac{d}{dx}f(x) - q(x) \cdot \frac{d}{dx}g(x)}{g(x)} \tag{1}$$

Since $q(x) = \frac{f(x)}{g(x)}$, therefore substituting it in equation (1) yields,

$$\frac{d}{dx}q(x) = \frac{\frac{d}{dx}f(x) - \frac{d}{dx}g(x) \cdot q(x)}{g(x)}$$

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$$
\begin{aligned}
&= \frac{\frac{d}{dx}f(x) - \frac{d}{dx}g(x) \cdot \frac{f(x)}{g(x)}}{g(x)} \\
&= \frac{\frac{d}{dx}f(x) - \frac{d}{dx}g(x) \cdot \frac{f(x)}{g(x)}}{g(x)} \cdot \underbrace{\frac{g(x)}{g(x)}}_{1} \\
&= \frac{g(x) \cdot \left(\frac{d}{dx}f(x) - \frac{d}{dx}g(x) \cdot \frac{f(x)}{g(x)}\right)}{g(x) \cdot g(x)} \\
&= \frac{\left(g(x)\frac{d}{dx}f(x) - \frac{d}{dx}g(x) \cdot \cancel{g(x)}\frac{f(x)}{\cancel{g(x)}}\right)}{(g(x))^2} \\
&= \frac{g(x)\frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{(g(x))^2} \\
\therefore \frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{g(x)\frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{(g(x))^2}
\end{aligned}
$$

This implies that the derivative of the quotient function $q(x) = \frac{f(x)}{g(x)}$ is $\frac{\cdot g(x)\frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{(g(x))^2}$. As a result, we have the following theorem.

Theorem 3.1.8. (The Quotient Rule) If $f(x)$ and $g(x)$ are both differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{\Big(g(x)\Big)^2}$$

Proof. We will prove this directly.

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{g(x+h)\cdot g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}
\end{aligned}
$$

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$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x)\overbrace{-f(x)\cdot g(x)+f(x)\cdot g(x)}^{0}-f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0} \frac{\dfrac{f(x+h)\cdot g(x)-f(x)\cdot g(x)+f(x)\cdot g(x)-f(x)\cdot g(x+h)}{h}}{g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0} \frac{\dfrac{f(x+h)\cdot g(x)-f(x)\cdot g(x)}{h}+\dfrac{f(x)\cdot g(x)-f(x)\cdot g(x+h)}{h}}{g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0} \frac{\left[g(x)\cdot\left(\dfrac{f(x+h)-f(x)}{h}\right)\right]-\left[f(x)\cdot\left(\dfrac{g(x+h)-g(x)}{h}\right)\right]}{g(x)\cdot g(x+h)} \\
&= \frac{\lim\limits_{h\to 0}\left[g(x)\cdot\left(\dfrac{f(x+h)-f(x)}{h}\right)\right]-\lim\limits_{h\to 0}\left[f(x)\cdot\left(\dfrac{g(x+h)-g(x)}{h}\right)\right]}{\lim\limits_{h\to 0}\Big(g(x)\cdot g(x+h)\Big)} \\
&= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}-\lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)} \\
&= \frac{g(x)\cdot\dfrac{d}{dx}f(x)-f(x)\cdot\dfrac{d}{dx}g(x)}{(g(x))^2}
\end{aligned}
$$

This completes the proof. ∎

Similar to the Product Rule, the Quotient Rule also may appear convoluted, thus it is often helpful to simplify it in the following equivalent manner.

Remark 3.1.2. (The Quotient Rule) If $f(x) = u$ and $g(x) = v$ are both differentiable functions, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\dfrac{d}{dx}(u)-u\cdot\dfrac{d}{dx}(v)}{(v)^2}$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.1.13.** What is the derivative of $f(x) = \dfrac{ex}{x}$?

Solution. Let $u = ex$ and $v = x$, then by the Quotient Rule,

$$\frac{d}{dx}\left(\frac{ex}{x}\right) = \frac{x\cdot\dfrac{d}{dx}ex-ex\cdot\dfrac{d}{dx}x}{x^2}$$

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$$
\begin{aligned}
&= \frac{x \cdot e\dfrac{dx}{dx} - ex \cdot \dfrac{dx}{dx}}{x^2} \\
&= \frac{ex - ex}{x^2} \\
&= \frac{0}{x^2} \\
&= 0
\end{aligned}
$$

Therefore, the derivative of $f(x) = \dfrac{ex}{x}$ is 0 ∎

► **Example 3.1.14.** Verify the result in the preceding example without using the Quotient Rule.

Solution. Notice that $f(x) = \dfrac{ex}{x} = e$. This suggests, by the constant rule, that

$$\frac{d}{dx}(e) = 0$$

Therefore, the derivative of $f(x) = \dfrac{ex}{x}$ is 0. This completes the verification. ∎

———————The End of Section 3.1

Elementary Rules of Differentiation

§ 3.2 Differentiation Rules for Trigonometric Functions

In this section, the derivatives of six trigonometric functions are derived. First, the derivatives of the sine and cosine functions are derived by the limit definition of derivative, which are then used to derive the derivatives of the remaining four trigonometric functions. Throughout this process, it is assumed that the variable x in $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ is measured in radians.

Chap. 3 / Sec. 3.2 / Subsec. 3.2.1 : Differentiation of $\sin(x)$

Among the other trigonometric functions, the sine function is one of the few that we will often encounter. This is because this function typically describes oscillations, waves, and circular motion, making it fundamental in both mathematics and physics. Therefore, we have the following theorem.

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Theorem 3.2.1. (The Derivative of the Sine Function) If $f(x) = \sin(x)$, then

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Prior to proving the theorem above, we first recall that if $f(x) = \sin(x)$, then $f(x+h)$ is given by the angle sum identity[1], that is,

$$\begin{aligned} f(x+h) &= \sin(x+h) \\ &= \sin(x)\cdot\cos(h) + \cos(x)\cdot\sin(h) \end{aligned}$$

and consider the following two limits that we had previously evaluated.

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{\cos(x)-1}{x} = 0$$

Proof. We will prove this directly.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\Big(\sin(x)\cdot\cos(h)+\cos(x)\cdot\sin(h)\Big)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)\cdot\cos(h)-\sin(x)+\cos(x)\cdot\sin(h)}{h} \\ &= \lim_{h\to 0}\frac{\Big[\sin(x)\cdot(\cos(h)-1)\Big]+\cos(x)\cdot\sin(h)}{h} \\ &= \lim_{h\to 0}\left[\frac{\Big[\sin(x)\cdot(\cos(h)-1)\Big]}{h}+\frac{\cos(x)\cdot\sin(h)}{h}\right] \\ &= \lim_{h\to 0}\left(\sin(x)\cdot\frac{\cos(h)-1}{h}\right)+\lim_{h\to 0}\left(\cos(x)\cdot\frac{\sin(h)}{h}\right) \\ &= \sin(x)\cdot\underbrace{\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right)}_{\text{the limit is 0}}+\cos(x)\cdot\underbrace{\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)}_{\text{the limit is 1}} \\ &= \sin(x)\cdot 0+\cos(x)\cdot 1 \\ &= \cos(x) \end{aligned}$$

This completes the proof. ∎

[1] $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$

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A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.1.** Differentiate $f(x) = x\sin(x)$.

Solution. Since $f(x)$ is a product of two functions, namely x and $\sin(x)$, therefore the Product Rule will be used as outlined below.

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\Big(x\sin(x)\Big)\\ &= x\cdot\frac{d}{dx}\sin(x) + \sin(x)\frac{d}{dx}x\\ &= x\cdot\cos(x) + \sin(x)\cdot 1\\ &= x\cos(x) + \sin(x)\end{aligned}$$

Therefore, the derivative of $f(x) = x\sin(x)$ is $f'(x) = x\cos(x) + \sin(x)$. ∎

► **Example 3.2.2.** Differentiate $f(x) = \dfrac{\sin(x)+1}{\sin(x)-1}$

Solution. Since $f(x)$ is a quotient of two functions $\sin(x)+1$ and $\sin(x)-1$, therefore the Quotient Rule will be used as outlined below.

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\left(\frac{\sin(x)+1}{\sin(x)-1}\right)\\ &= \frac{(\sin(x)-1)\cdot\frac{d}{dx}(\sin(x)+1) - (\sin(x)+1)\cdot\frac{d}{dx}(\sin(x)-1)}{(\sin(x)-1)^2}\\ &= \frac{(\sin(x)-1)\cdot\frac{d}{dx}\sin(x) + \frac{d}{dx}(1) - (\sin(x)+1)\cdot\frac{d}{dx}\sin(x) - \frac{d}{dx}(1)}{(\sin(x)-1)^2}\\ &= \frac{(\sin(x)-1)\cdot\cos(x) + 0 - (\sin(x)+1)\cdot\cos(x) - 0}{(\sin(x)-1)^2}\\ &= \frac{(\sin(x)-1)\cdot\cos(x) - (\sin(x)+1)\cdot\cos(x)}{(\sin(x)-1)^2}\\ &= \frac{\sin(x)\cos(x) - \cos(x) - \sin(x)\cos(x) - \cos(x)}{(\sin(x)-1)^2}\\ &= \frac{-\cos(x) - \cos(x)}{(\sin(x)-1)^2}\\ &= -\frac{2\cos(x)}{(\sin(x)-1)^2}\end{aligned}$$

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Therefore, the derivative of $f(x) = \dfrac{\sin(x)+1}{\sin(x)-1}$ is $f'(x) = -\dfrac{2\cos(x)}{(\sin(x)-1)^2}$. ∎

Now that the derivative of $\sin(x)$ is determined, we move to find the derivative of $\cos(x)$.

Chap. 3 / Sec. 3.2 / Subsec. 3.2.2 : Differentiation of $\cos(x)$

In the previous section, it is observed that the derivative of the sine function is the cosine function. Now, it will be shown, surprisingly, that the derivative of the cosine function is the negative of the sine function. That is,

Theorem 3.2.2. (The Derivative of the Cosine Function) If $f(x) = \cos(x)$, then

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Now, prior to proving the theorem above by the definition of derivative, it is important to recall that if $f(x) = \cos(x)$, then $f(x+h)$ is given by the angle sum identity[2] as follows,

$$\begin{aligned} f(x+h) &= \cos(x+h) \\ &= \cos(x)\cdot\cos(h) - \sin(x)\cdot\sin(h) \end{aligned}$$

Proof. We will prove this directly.

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\ &= \lim_{h\to 0}\frac{\Big(\cos(x)\cdot\cos(h)-\sin(x)\cdot\sin(h)\Big)-\cos(x)}{h} \\ &= \lim_{h\to 0}\frac{\cos(x)\cdot\cos(h)-\cos(x)-\sin(x)\cdot\sin(h)}{h} \\ &= \lim_{h\to 0}\frac{\Big[\cos(x)\cdot\Big(\cos(h)-1\Big)\Big]-\sin(x)\cdot\sin(h)}{h} \\ &= \lim_{h\to 0}\left[\frac{[\cos(x)\cdot(\cos(h)-1)]}{h}-\frac{\sin(x)\cdot\sin(h)}{h}\right] \\ &= \lim_{h\to 0}\left(\cos(x)\cdot\frac{\cos(h)-1}{h}\right)-\lim_{h\to 0}\left(\sin(x)\cdot\frac{\sin(h)}{h}\right) \end{aligned}$$

[2] $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

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$$
\begin{aligned}
&= \cos(x) \cdot \underbrace{\left(\lim_{h\to 0} \frac{\cos(h)-1}{h}\right)}_{\text{the limit is } 0} - \sin(x) \cdot \underbrace{\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)}_{\text{the limit is } 1} \\
&= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\
&= -\sin(x)
\end{aligned}
$$

This completes the proof. ∎

A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.3.** Differentiate $f(x) = \cos^{\cos(0)}(x)$

Solution. Since $\cos(0) = 1$, therefore,

$$
\begin{aligned}
\frac{d}{dx}\cos^{\cos(0)}(x) &= \frac{d}{dx}\cos^{1}(x) \\
&= -\sin(x)
\end{aligned}
$$

Therefore, the derivative of $f(x) = \cos^{\cos(0)}(x)$ is $f'(x) = -\sin(x)$. ∎

► **Example 3.2.4.** Differentiate $f(x) = x^2\cos(x)$

Solution. Since $f(x)$ is a product of two functions x^2 and $\cos(x)$, thus the Product Rule will be used as follows.

$$
\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\Big(x^2\cos(x)\Big) \\
&= x^2 \cdot \frac{d}{dx}\cos(x) + \cos(x) \cdot \frac{d}{dx}\Big(x^2\Big) \\
&= x^2 \cdot \Big(-\sin(x)\Big) + \cos(x) \cdot 2x \\
&= -x^2\sin(x) + 2x\cos(x) \\
&= x\Big(2\cos(x) - x\sin(x)\Big)
\end{aligned}
$$

Therefore, the derivative of $f(x) = x^2\cos(x)$ is $f'(x) = x\Big(2\cos(x) - x\sin(x)\Big)$. ∎

Chap. 3 / Sec. 3.2 / Subsec. 3.2.3 : Differentiation of $\tan(x)$

Now that we've obtained the derivatives of $\sin(x)$ and $\cos(x)$, we can use them to derive the derivatives of other trigonometric functions. In order to derive the

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derivative of $\tan(x)$, recall that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

This implies that,

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\cos(x)\cdot\frac{d}{dx}\sin(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

This implies that the derivative of the tangent function is the secant function squared.

Theorem 3.2.3. (The Derivative of the Tangent Function) If $f(x) = \tan(x)$, then

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.5.** Differentiate $f(x) = \tan(x) + x$.

Solution. By the Summation Rule, we get,

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\Big(\tan(x) + x\Big) \\
&= \frac{d}{dx}\Big(\tan(x)\Big) + \frac{d}{dx}(x) \\
&= \sec^2(x) + 1
\end{aligned}$$

Therefore, the derivative of $f(x) = \tan(x) + x$ is $f'(x) = \sec^2(x) + 1$. ∎

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► **Example 3.2.6.** Differentiate $f(x) = \dfrac{\sin^2(x)}{\cos(x)}$

Solution. Since $\dfrac{\sin^2(x)}{\cos(x)}$ can be simplified as follows,

$$\begin{aligned}\frac{\sin^2(x)}{\cos(x)} &= \frac{\sin(x)}{\cos(x)} \cdot \sin(x)\\ &= \tan(x) \cdot \sin(x)\end{aligned}$$

Therefore, by the Product Rule,

$$\begin{aligned}\frac{d}{dx}\left(\tan(x) \cdot \sin(x)\right) &= \tan(x) \cdot \frac{d}{dx}\sin(x) + \sin(x) \cdot \frac{d}{dx}\tan(x)\\ &= \tan(x) \cdot \cos(x) + \sin(x) \cdot \sec^2(x)\\ &= \frac{\sin(x)}{\cancel{\cos(x)}} \cdot \cancel{\cos(x)} + \sin(x) \cdot \sec^2(x)\\ &= \sin(x)(1 + \sec^2(x))\end{aligned}$$

Therefore, the derivative of $f(x) = \dfrac{\sin^2(x)}{\cos(x)}$ is $f'(x) = \sin(x)(1 + \sec^2(x))$. ∎

Chap. 3 / Sec. 3.2 / Subsec. 3.2.4 : Differentiation of $\cot(x)$

Following a similar method as before, the derivative of $\cot(x)$ is determined. For that, recall that

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

With that in mind, we proceed as follows,

$$\begin{aligned}\frac{d}{dx}\cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)\\ &= \frac{\sin(x) \cdot \dfrac{d}{dx}\cos(x) - \cos(x) \cdot \dfrac{d}{dx}\sin(x)}{\sin^2(x)}\\ &= \frac{\sin(x) \cdot (-\sin(x)) - \cos(x) \cdot \cos(x)}{\sin^2(x)}\\ &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}\\ &= \frac{(-1) \cdot \left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)}\end{aligned}$$

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$$
\begin{aligned}
&= \frac{(-1)\cdot(1)}{\sin^2(x)} \\
&= -\frac{1}{\sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}
$$

This implies that the derivative of the cotangent function is the negative of the cosecant function squared. That is,

Theorem 3.2.4. (The Derivative of the Cotangent Function) If $f(x) = \cot(x)$, then

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.7.** Differentiate $f(x) = 2x + 2\cot(x)$

Solution. By the Summation Rule, we get,

$$
\begin{aligned}
\frac{d}{dx}\left(2x + 2\cot(x)\right) &= \frac{d}{dx}2x + \frac{d}{dx}2\cot(x) \\
&= 2\frac{dx}{dx} + 2\frac{d}{dx}\cot(x) \\
&= 2 - 2\csc^2(x)
\end{aligned}
$$

Therefore, the derivative of $f(x) = 2x + 2\cot(x)$ is $f'(x) = 2 - 2\csc^2(x)$. ∎

► **Example 3.2.8.** Differentiate $f(x) = \dfrac{\cos^2(x)}{e\sin(x)}$

Solution. Since $\dfrac{\cos^2(x)}{e\sin(x)}$ can be simplified as follows,

$$
\begin{aligned}
\frac{\cos^2(x)}{e\sin(x)} &= \frac{1}{e}\cdot\frac{\cos(x)}{\sin(x)}\cdot\cos(x) \\
&= \frac{1}{e}\cot(x)\cdot\cos(x)
\end{aligned}
$$

Therefore,

$$\frac{d}{dx}\left(\frac{1}{e}\cot(x)\cdot\cos(x)\right) = \frac{1}{e}\cdot\frac{d}{dx}\left(\cot(x)\cdot\cos(x)\right)$$

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$$
\begin{aligned}
&= \frac{1}{e}\left(\cot(x)\cdot\frac{d}{dx}\cos(x) + \cos(x)\cdot\frac{d}{dx}\cot(x)\right)\\
&= \frac{1}{e}\left(\cot(x)\cdot(-\sin(x)) + \cos(x)\cdot(-\csc^2(x))\right)\\
&= \frac{1}{e}\left(-\sin(x)\cot(x) - \cos(x)\csc^2(x)\right)\\
&= \frac{-\cancel{\sin(x)}\dfrac{\cos(x)}{\cancel{\sin(x)}} - \cos(x)\csc^2(x)}{e}\\
&= \frac{-\cos(x) - \cos(x)\csc^2(x)}{e}
\end{aligned}
$$

Therefore, the derivative of $f(x) = \dfrac{\cos^2(x)}{e\sin(x)}$ is $f'(x) = \dfrac{-\cos(x) - \cos(x)\csc^2(x)}{e}$ ∎

Chap. 3 / Sec. 3.2 / Subsec. 3.2.5 : Differentiation of $\sec(x)$

To derive the derivative of $f(x) = \sec(x)$, notice that since $\sec(x)$ can be written as $\dfrac{1}{\cos(x)}$, therefore, by the Quotient Rule,

$$
\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)\\
&= \frac{\cos(x)\cdot\dfrac{d}{dx}(1) - 1\cdot\dfrac{d}{dx}\cos(x)}{\cos^2(x)}\\
&= \frac{\cos(x)\cdot 0 - 1\cdot\Big(-\sin(x)\Big)}{\cos^2(x)}\\
&= \frac{\sin(x)}{\cos^2(x)}\\
&= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\\
&= \sec(x)\cdot\tan(x)
\end{aligned}
$$

This implies that the derivative of the secant function is the secant function multiplied by the tangent function. That is,

Theorem 3.2.5. (The Derivative of the Secant Function) If $f(x) = \sec(x)$,

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then

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.9.** Differentiate $f(x) = x^{2\cos(0)} - 2\cos(0)\cdot\sec(x)$

Solution. Since $\cos(0) = 1$, therefore, $f(x)$ is simplified into the following,

$$x^{2\cos(0)} - 2\cos(0)\cdot\sec(x) = x^2 - 2\sec(x)$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\left(x^2 - 2\sec(x)\right) &= \frac{d}{dx}x^2 - \frac{d}{dx}2\sec(x)\\ &= 2x - 2\sec(x)\tan(x)\end{aligned}$$

Therefore, the derivative of $f(x) = x^{2\cos(0)} - 2\cos(0)\cdot\sec(x)$ is $2x-2\sec(x)\tan(x)$. ∎

► **Example 3.2.10.** Differentiate $f(x) = \sin(x)\sec(x)$

Solution. By the Product Rule,

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)\sec(x)\right) &= \sin(x)\cdot\frac{d}{dx}\sec(x) + \sec(x)\cdot\frac{d}{dx}\sin(x)\\ &= \sin(x)\cdot\sec(x)\tan(x) + \sec(x)\cdot\cos(x)\\ &= \sin(x)\cdot\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} + \frac{1}{\cancel{\cos(x)}}\cdot\cancel{\cos(x)}\\ &= \frac{\sin^2(x)}{\cos^2(x)} + 1\\ &= \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

Therefore, the derivative of $f(x) = \sin(x)\sec(x)$ is $f'(x) = \sec^2(x)$. ∎

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Chap. 3 / Sec. 3.2 / Subsec. 3.2.6 : Differentiation of $\csc(x)$

To derive the derivative of $f(x) = \csc(x)$, notice that $\csc(x)$ can be written as $\dfrac{1}{\sin(x)}$, therefore, by the Quotient Rule,

$$\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{\sin(x)\cdot\dfrac{d}{dx}(1) - 1\cdot\dfrac{d}{dx}\sin(x)}{\sin^2(x)} \\
&= \frac{\sin(x)\cdot 0 - 1\cdot\cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cdot\cot(x)
\end{aligned}$$

This implies that the derivative of the cosecant function is the negative of the cosecant function multiplied by the cotangent function. That is,

Theorem 3.2.6. (The Derivative of the Cosecant Function) If $f(x) = \csc(x)$, then

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.2.11.** Differentiate $f(x) = \pi x - \csc(x)$

Solution. By the difference rule,

$$\begin{aligned}
\frac{d}{dx}\left(\pi x - \csc(x)\right) &= \pi\frac{d}{dx}x - \frac{d}{dx}\csc(x) \\
&= \pi + \csc(x)\cot(x)
\end{aligned}$$

Therefore, the derivative of $f(x) = \pi x - \csc(x)$ is $f'(x) = \pi + \csc(x)\cot(x)$. ∎

► **Example 3.2.12.** Differentiate $f(x) = \csc^2(x)$

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Solution. By the Product Rule, we get,

$$\begin{aligned}\frac{d}{dx}\csc^2(x) &= \frac{d}{dx}\left(\csc(x)\cdot\csc(x)\right)\\ &= \csc(x)\cdot\frac{d}{dx}\csc(x)+\csc(x)\cdot\frac{d}{dx}\csc(x)\\ &= \csc(x)\cdot\left(-\csc(x)\cot(x)\right)+\csc(x)\cdot\left(-\csc(x)\cot(x)\right)\\ &= -\csc^2(x)\cot(x)-\csc^2(x)\cot(x)\\ &= -2\csc^2(x)\cot(x)\end{aligned}$$

Therefore, the derivative of $f(x)=\csc^2(x)$ is $f'(x)=-2\csc^2(x)\cot(x)$. ∎

———————The End of Section 3.2

Differentiation Rules for Trigonometric Functions

§ 3.3 Differentiation of Composite Functions (The Chain Rule)

The differentiation rules covered so far are useful for finding the derivatives of many basic functions. However, these rules alone are often not enough when we encounter more complex functions that are frequently used in practice, such as composite functions. To handle such demanding cases, we need some handy tools such as the Chain Rule, which allow us to differentiate functions that are a composition of others. Take the following function as an example,

$$y=\left(x^4+14\right)^3 \tag{1}$$

To differentiate this function using the rules discussed so far, we must first need to expand it and then apply the appropriate differentiation rules. For example, since,

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3 \tag{2}$$

Therefore, applying (2) to (1) yields,

$$y=x^{12}+42x^8+588x^4+2744 \tag{3}$$

We can now differentiate (3) using the Power and Summation Rule. That is,

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(x^{12}+42x^8+588x^4+2744\right)$$

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$$
\begin{aligned}
&= \frac{d}{dx}\left(x^{12} + 42x^8 + 588x^4 + 2744\right) \\
&= \frac{d}{dx}\left(x^{12}\right) + \frac{d}{dx}\left(42x^8\right) + \frac{d}{dx}\left(588x^4\right) + \frac{d}{dx}\left(2744\right) \\
&= 12x^{11} + 336x^7 + 2352x^3 \qquad (4)
\end{aligned}
$$

Thus, we've derived the derivative of $y = \left(x^4 + 14\right)^3$ to be $12x^{11} + 336x^7 + 2352x^3$. Now, can the same approach be taken to find the derivative of $y = \left(x^4 + 14\right)^{235711131719}$? The answer is quite obviously no. Therefore, let us begin to develop a new method to tackle such situations.

Notice that the function $y = \left(x^4 + 14\right)^3$ is a composite function. That is, it is composed of two different functions in a way that if we let a variable u to stand for the expression $x^4 + 14$, then $y = \left(x^4 + 14\right)^3$ becomes $y = u^3$, where $u = x^4 + 14$. This implies that y is now a function of u (as $y = u^3$), and u is in turn a function of x (as $u = x^4 + 14$).

Since, y is a function of u, therefore the derivative (i.e., the rate of change) of y with respect to u is,

$$
\begin{aligned}
\frac{dy}{du} &= \frac{d}{du}(u^3) \\
&= 3u^2
\end{aligned}
$$

Now, since u in turn is a function of x, therefore the derivative (i.e., the rate of change) of u with respect to x is,

$$
\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\left(x^4 + 14\right) \\
&= \frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(14\right) \\
&= 4x^3 + 0 \\
&= 4x^3
\end{aligned}
$$

Now, notice carefully that,

If the output y changes $3u^2$ times as fast as the input u, since $y = u^3$

And, if the output u changes $4x^3$ times as fast as the input x, since $y = x^4 + 14$

Then, the output y changes $\left(3u^2\right) \times \left(4x^3\right)$ times as fast as x, for $u = \left(x^4 + 14\right)^3$

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In notations,

$$\begin{aligned} \text{if} \quad & \frac{dy}{du} = 3u^2 \\ \text{and if} \quad & \frac{du}{dx} = 4x^3 \\ \text{then} \quad & \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(3u^2\right) \times \left(4x^3\right) \end{aligned}$$

Therefore, the derivative of $y = \left(x^4 + 14\right)^3$ with respect to x is,

$$\begin{aligned} 3u^2 \cdot 4x^3 &= 3\left(x^4 + 14\right)^2 \cdot 4x^3 \\ &= 12x^3\left(x^4 + 14\right)^2 \end{aligned}$$

Which is indeed equivalent to equation (4) when expanded.

Now, we try to find the derivative of the challenging function $y = \left(x^4 + 14\right)^{235711131719}$ using the method described above. For that, let $u = x^4 + 14$, then $y = u^{235711131719}$. Since y is a function of u, therefore, it is differentiated with respect to u,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du}\left(u^{235711131719}\right) \\ &= 235711131719u^{235711131719-1} \\ &= 235711131719u^{235711131718} \end{aligned}$$

Since $u = x^4 + 14$ is a function of x, therefore, it is differentiated with respect to x,

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}\left(x^4 + 14\right) \\ &= \frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(14\right) \\ &= 4x^3 + 0 \\ &= 4x^3 \end{aligned}$$

Now, the derivative of y with respect to x (not u) happens to be:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 235711131719u^{235711131718} \times 4x^3 \\ &= 235711131719\left(x^4 + 14\right)^{235711131718} \cdot 4x^3 \end{aligned}$$

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$$= 942844526876x^3\left(x^4+14\right)^{235711131718}$$

Summarizing the discussion, we have the following theorem to differentiate composite functions.

Theorem 3.3.1. (The Chain Rule) If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the derivative of the composite function $y = f(g(x))$ with respect to x is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$
$$y' = f'(u) \cdot u'$$

We here provide an *intuitive* argument of the Chain Rule. Note that it isn't sufficiently rigorous to be accepted as a complete proof of the theorem.

Intuitive Argument. Since u is a function of x, therefore any change Δx in x produces a corresponding change Δu in u. This, in turn, produces a corresponding change Δy in y, as y is a function of u. Now, consider the following,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} && \text{by the definition of derivative}\\
&= \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\cdot 1\right)\\
&= \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\cdot\frac{\Delta u}{\Delta u}\right)\\
&= \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}\right)\\
&= \left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\right)\cdot\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right)
\end{aligned}$$

Since, differentiability implies continuity, therefore $\Delta u \to 0$ as $\Delta x \to 0$.

$$\begin{aligned}
&= \left(\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\right)\cdot\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right)\\
&= \frac{dy}{du}\cdot\frac{du}{dx}
\end{aligned}$$

This completes the argument intuitively. ∎

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Remark 3.3.1. When applying the Chain rule, it is often helpful to think of the composite function $y = f\Big(g(x)\Big)$ as having two parts. One is the *inner part* $u = g(x)$ and the other is the *outer part* $y = f(u)$. This means that to differentiate a composite function, we:

1. Differentiate the outer function $y = f(u)$ with respect to the inner function $u = g(x)$.
2. Multiply the result by the derivative of the inner function $u = g(x)$ with respect to x.

To better understand the inner and outer function analogy, consider the following functions.

$$\sin(x^2) \quad \text{and} \quad \sin^2(x)$$

Although the inner and outer parts of these functions may resemble one another, they are fundamentally different. The inner part of the first function is $u = x^2$, and the outer part of it is $y = \sin(u)$. Whereas the inner part of the second function is $u = \sin(x)$, and the outer part of it is $y = u^2$, since $\sin^2(x) = \Big(\sin(x)\Big)^2$.

► **Example 3.3.1.** Differentiate $f(x) = \sin(x^2)$

Solution. For the function $\sin(x^2)$, the inner part u is x^2 and the outer part y is $\sin(u)$.
Since the outer part $y = \sin(u)$ is a function of u, therefore it is differentiated with respect to u,

$$\begin{aligned}\frac{dy}{du} &= \frac{d}{du}\sin(u)\\ &= \cos(u)\end{aligned}$$

Similarly, the inner part $u = x^2$ is a function of x, therefore it is differentiated with respect to x,

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}(x^2)\\ &= 2x\end{aligned}$$

Now, by the Chain Rule, we differentiate y with respect to x as follows,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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$$
\begin{aligned}
&= \cos(u) \cdot 2x \\
&= \cos(x^2) \cdot 2x \\
&= 2x \cos(x^2)
\end{aligned}
$$

Therefore, the derivative of $f(x) = \sin(x^2)$ is $f'(x) = 2x\cos(x^2)$. ∎

► **Example 3.3.2.** Differentiate $f(x) = \sin^2(x)$

Solution. For the function $\sin^2(x) = \Big(\sin(x)\Big)^2$, the inner part u is $\sin(x)$ and the outer part y is u^2.
Since the outer part $y = u^2$ is a function of u, therefore it is differentiated with respect of u,

$$
\begin{aligned}
\frac{dy}{du} &= \frac{d}{du}\Big(u^2\Big) \\
&= 2u
\end{aligned}
$$

Similarly, the inner part $u = \sin(x)$ is a function of x, therefore it is differentiated with respect to x,

$$
\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\Big(\sin(x)\Big) \\
&= \cos(x)
\end{aligned}
$$

Now, by the Chain Rule, we differentiate y with respect to x as follows,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= 2u \cdot \cos(x) \\
&= 2\sin(x)\cos(x)
\end{aligned}
$$

Therefore, the derivative of $f(x) = \sin^2(x)$ is $f'(x) = 2\sin(x)\cos(x)$. ∎

Author's Interruption. In practicality, it is often unnecessary to write out all the details of the Chain Rule. Instead, one can directly apply the Chain Rule without explicitly mentioning the inner and outer function.

To demonstrate this, we apply the Chain Rule directly to the function shown in Example 3.3.1, as follows,

$$
\frac{dy}{dx} = \frac{d}{dx}\sin\left(x^2\right)
$$

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$$
\begin{aligned}
&= \cos\left(x^2\right) \cdot \frac{d}{dx}\left(x^2\right) \\
&= \cos\left(x^2\right) \cdot 2x \\
&= 2x\cos\left(x^2\right)
\end{aligned}
$$

That is, the outermost function is first differentiated and then simply multiplied by the derivative of the inner function.

► **Example 3.3.3.** Differentiate $y = \sqrt{x^2 + x}$

Solution. For the function $y = \sqrt{x^2 + x}$, the inner and outer parts are $u = x^2 + x$ and $y = \sqrt{u}$, respectively. Therefore, by the Chain Rule,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\left(\sqrt{u}\right) \cdot \frac{d}{dx}\left(x^2 + x\right) \\
&= \frac{1}{2\sqrt{u}} \cdot 2x + 1 \\
&= \frac{2x + 1}{2\sqrt{u}} \\
&= \frac{2x + 1}{2\sqrt{x^2 + x}}
\end{aligned}
$$

Therefore, the derivative of $f(x) = \sqrt{x^2 + x}$ is $f'(x) = \dfrac{2x + 1}{2\sqrt{x^2 + x}}$. ∎

► **Example 3.3.4.** Differentiate $y = \left(\dfrac{x^2}{x + 1}\right)^2$

Solution. For the function $y = \left(\dfrac{x^2}{x + 1}\right)^2$, the inner and outer parts are $u = \dfrac{x^2}{x + 1}$ and u^2, respectively. Therefore, by the Chain Rule,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\left(u^2\right) \cdot \frac{d}{dx}\left(\frac{x^2}{x + 1}\right) \\
&= 2u \cdot \frac{(x + 1) \cdot \dfrac{d}{dx}\left(x^2\right) - x^2 \cdot \dfrac{d}{dx}\left(x + 1\right)}{(x + 1)^2}
\end{aligned}
$$

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$$
\begin{aligned}
&= 2u \cdot \frac{(x+1)\cdot 2x - x^2 \cdot 1}{(x+1)^2} \\
&= 2u \cdot \frac{2x(x+1) - x^2}{(x+1)^2} \\
&= 2\left(\frac{x^2}{x+1}\right) \cdot \frac{2x(x+1) - x^2}{(x+1)^2} \\
&= \frac{2x^2}{x+1} \cdot \frac{2x^2 + 2x - x^2}{(x+1)^2} \\
&= \frac{2x^2\left(2x^2 + 2x - x^2\right)}{(x+1)^3} \\
&= \frac{4x^4 + 4x^3 - 2x^4}{(x+1)^3} \\
&= \frac{2x^3(2x + 2 - x)}{(x+1)^3} \\
&= \frac{2x^3(x+2)}{(x+1)^3}
\end{aligned}
$$

Therefore, the derivative of $f(x) = \left(\frac{x^2}{x+1}\right)^2$ is $f'(x) = \frac{2x^3(x+2)}{(x+1)^3}$.

Note that if the equation were to be simplified as $\left(\frac{x^2}{x+1}\right)^2 = \frac{x^4}{(x+1)^2}$ prior to differentiating it solely using the Quotient Rule, the result would've remained the same. ∎

Since the Chain Rule is one of the most important and frequently encountered rule among the differentiation rules, therefore a few additional examples are given below.

► **Example 3.3.5.** Differentiate $y = \sqrt{\sin(x)}$

Solution. The derivative of $f(x) = \sqrt{\sin(x)}$ is,

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\sqrt{\sin(x)} \\
&= \frac{d}{dx}\sin(x)^{\frac{1}{2}} \\
&= \frac{1}{2}\sin(x)^{-\frac{1}{2}} \cdot \frac{d}{dx}\sin(x) \\
&= \frac{1}{2} \cdot \frac{1}{\sin(x)^{\frac{1}{2}}} \cdot \cos(x)
\end{aligned}
$$

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$$= \frac{\cos(x)}{2\sqrt{\sin(x)}}$$

Therefore, the derivative of $f(x) = \sqrt{\sin(x)}$ is $f'(x) = \dfrac{\cos(x)}{2\sqrt{\sin(x)}}$. ∎

► **Example 3.3.6.** Differentiate $y = \sin\left(\cos\left(x^2\right)\right)$

Solution. Applying the Chain Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin\left(\cos\left(x^2\right)\right) \\
&= \cos\left(x^2\right) \cdot \frac{d}{dx}\cos\left(x^2\right) \\
&= \cos\left(x^2\right) \cdot -\sin\left(x^2\right) \cdot \frac{d}{dx}\left(x^2\right) \\
&= -\cos\left(x^2\right) \cdot \sin\left(x^2\right) \cdot 2x \\
&= -2x\cos\left(x^2\right)\sin\left(x^2\right)
\end{aligned}$$

Therefore, the derivative of $f(x) = \sin\left(\cos\left(x^2\right)\right)$ is $f'(x) = -2x\cos\left(x^2\right)\sin\left(x^2\right)$. ∎

► **Example 3.3.7.** Differentiate $y = \sin\left(\cos\left(\tan\left(x^2\right)\right)\right)$.

Solution. Applying the Chain Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin\left(\cos\left(\tan\left(x^2\right)\right)\right) \\
&= \cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot \frac{d}{dx}\left[\cos\left(\tan\left(x^2\right)\right)\right] \\
&= \cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot -\sin\left(\tan\left(x^2\right)\right) \cdot \frac{d}{dx}\tan\left(x^2\right) \\
&= \cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot -\sin\left(\tan\left(x^2\right)\right) \cdot \sec^2\left(x^2\right) \cdot \frac{d}{dx}\left(x^2\right) \\
&= -\cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot \sin\left(\tan\left(x^2\right)\right) \cdot \sec^2\left(x^2\right) \cdot 2x \\
&= -2x \cdot \cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot \sin\left(\tan\left(x^2\right)\right) \cdot \sec^2\left(x^2\right)
\end{aligned}$$

Therefore, the derivative of $f(x) = \sin\left(\cos\left(\tan\left(x^2\right)\right)\right)$ is $f'(x) = -2x \cdot \cos\left(\cos\left(\tan\left(x^2\right)\right)\right) \cdot \sin\left(\tan\left(x^2\right)\right) \cdot \sec^2\left(x^2\right)$. ∎

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► **Example 3.3.8.** Differentiate $y = \sin\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right)$

Solution. Applying the Chain Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot \frac{d}{dx}\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \frac{d}{dx}\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot \frac{d}{dx}\cot\left(\sec\left(\csc\left(x^2\right)\right)\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot -\csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right) \cdot \frac{d}{dx}\sec\left(\csc\left(x^2\right)\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot -\csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right) \cdot \sec\left(\csc\left(x^2\right)\right)\tan\left(\csc\left(x^2\right)\right) \\
&\quad \cdot \frac{d}{dx}\csc\left(x^2\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot -\csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right) \cdot \sec\left(\csc\left(x^2\right)\right)\tan\left(\csc\left(x^2\right)\right) \\
&\quad \cdot -\csc\left(x^2\right)\cot\left(x^2\right) \cdot \frac{d}{dx}\left(x^2\right) \\
&= \cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot -\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot -\csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right) \cdot \sec\left(\csc\left(x^2\right)\right)\tan\left(\csc\left(x^2\right)\right) \\
&\quad \cdot -\csc\left(x^2\right)\cot\left(x^2\right) \cdot 2x \\
&= -2x\cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right)\sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \\
&\quad \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right)\sec\left(\csc\left(x^2\right)\right)\tan\left(\csc\left(x^2\right)\right) \\
&\quad \csc\left(x^2\right)\cot\left(x^2\right)
\end{aligned}$$

Therefore, the derivative of $f(x) = \sin\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right)$ is $f'(x) = -2x\cos\left(\cos\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right)\right) \cdot \sin\left(\tan\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right)\right) \cdot \sec^2\left(\cot\left(\sec\left(\csc\left(x^2\right)\right)\right)\right) \cdot \csc^2\left(\sec\left(\csc\left(x^2\right)\right)\right) \cdot \sec\left(\csc\left(x^2\right)\right) \cdot \tan\left(\csc\left(x^2\right)\right) \cdot \csc\left(x^2\right)\cot\left(x^2\right)$. ∎

THE END OF SECTION 3.3

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Differentiation of Composite Functions
(The Chain Rule)

§ 3.4 Implicit Differentiation

As of yet, all the functions that have been differentiated were of the form,

$$y = f(x)$$

Where, y is a function defined explicitly of x, as it is expressed entirely in terms of x. However, times do occur when y is not expressed explicitly in terms of x, while still being a function of x. For example, consider the following equation,

$$xy - 1 = 0$$

The equation above defines y implicitly as a function of x. This is because y can be expressed in terms of x if we rearrange the equation as follows.

$$\begin{aligned} xy - 1 &= 0 \\ xy &= 1 \\ y &= \frac{1}{x} \end{aligned}$$

It is now evident—after rearrangement of the equation—that y is indeed a function of x, even though it was initially defined implicitly by the equation $xy - 1 = 0$. Such a function, where y is defined indirectly by an equation, is called an *implicit function* of x.

In general, if a relationship between x and y is given by an equation of the form,

$$F(x, y) = 0 \tag{1}$$

where F is a function of both x and y, and y is not explicitly solved in terms of x, then y is said to be an implicit function of x.

In contrast to our intuition, an implicit equation may not always define a single function. When the graph of an implicit equation fails the vertical line test, the equation defines more than one function. For example, consider the following equation,

$$x^2 + y^2 = 1$$

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The graph of this equation is the graph of a unit circle that fails the vertical line test. Solving this for y yields,

$$y = \pm\sqrt{1 - x^2}$$

where, $y = +\sqrt{1 - x^2}$ represents the upper semicircle and $y = -\sqrt{1 - x^2}$ represents the lower semicircle of the circle $x^2 + y^2 = 1$.

With that in mind, notice now that some equations defining implicit functions y are notoriously difficult—if not impossible—to express in terms of x explicitly. And, so far, we have seen that in order to differentiate a function, it needs to be in the form of $y = f(x)$. So, how such implicitly defined functions be differentiated? Is it even possible?

The answer is positive. In general, it is not required to explicitly solve an equation for y in terms of x to differentiate implicitly defined functions. For example, consider the previous function,

$$\begin{aligned} xy - 1 &= 0 \\ xy &= 1 \end{aligned} \tag{2}$$

The typical approach to differentiating this equation is to first rewrite the equation as $y = \dfrac{1}{x}$ and then to differentiate y with respect to x, as shown below.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \frac{d}{dx}\left(x^{-1}\right) \\ &= -1 \cdot x^{-1-1} \\ &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

Now, instead of this typical method, if we consider y as a differentiable function of x, we can differentiate equation (2) directly as follows.

$$\begin{aligned} \frac{d}{dx}\Big(xy\Big) &= \frac{d}{dx}(1) \\ x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x) &= 0 \\ x \cdot \frac{dy}{dx} + y &= 0 \end{aligned} \tag{3}$$

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Now, solving for $\frac{dy}{dx}$ algebraically yields,

$$\begin{aligned} x \cdot \frac{dy}{dx} &= -y \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Thus, the derivative of the implicit function $xy = 1$ is $-\frac{y}{x}$. Notice that the derivative, in this case, is expressed in terms of both x and y, instead of just x. So, to evaluate the derivative at a specific point, both values of x and y are required.

Additionally, notice how the Chain Rule is used in equation (3). That is, when differentiating terms with respect to x that contain only the variable x, the process follows the usual method. However, when we encounter terms involving variables other than x, the Chain Rule is applied there.

The derivative obtained here is indeed equivalent to the derivative obtained by the typical method if $y = \frac{1}{x}$ is substituted, as shown below.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{y}{x} \\ &= -\frac{\frac{1}{x}}{x} = -\frac{1}{x} \cdot \frac{1}{x} = -\frac{1}{x^2} \end{aligned}$$

The method by which the derivative of $xy = 1$ is obtained without having to solve for y in terms of x, is called *implicit differentiation*. Implicit differentiation is a technique used to find the derivative of functions when the relationship between their variables are not explicitly given in the form $y = f(x)$. Instead, the variables are interconnected by an equation such as $F(x, y) = 0$. In such cases, y is treated as a function of x and the rules of differentiation are applied to both sides of the equation, differentiating y implicitly.

In general, implicit differentiation has the following steps.

Remark 3.4.1.

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving $\frac{dy}{dx}$ on one side of the equation and move

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all the remaining terms to the other side.

3. Factor out $\frac{dy}{dx}$ when necessary.
4. Solve for $\frac{dy}{dx}$.

With that in mind, we will now consider a few examples to understand the implicit differentiation.

► **Example 3.4.1.** Differentiate $x^2 + y^3 = 5$.

Solution. By implicit differentiation

$$\begin{aligned} x^2 + y^3 &= 5 \\ \frac{d}{dx}\left(x^2 + y^3\right) &= \frac{d}{dx}(5) \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^3 &= 0 \\ 2x + 3y^2 \cdot \frac{dy}{dx} &= 0 \\ 3y^2 \cdot \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{2x}{3y^2} \end{aligned}$$

Therefore, the derivative of $x^2 + y^3 = 5$ is $\frac{dy}{dx} = -\frac{2x}{3y^2}$. ∎

► **Example 3.4.2.** Differentiate $x^2 + xy + y^3 = x$.

Solution. By implicit differentiation

$$\begin{aligned} \frac{d}{dx}\left(x^2 + xy + y^3\right) &= \frac{d}{dx}(x) \\ \frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^3 &= \frac{dx}{dx} \\ 2x + \left(x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx}\right) + 3y^2 \cdot \frac{dy}{dx} &= 1 \\ 2x + x \cdot \frac{dy}{dx} + y + 3y^2 \cdot \frac{dy}{dx} &= 1 \\ x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} &= 1 - 2x - y \end{aligned}$$

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$$\frac{dy}{dx}\left(x+3y^2\right) = 1-2x-y$$
$$\frac{dy}{dx} = \frac{1-2x-y}{x+3y^2}$$

Therefore, the derivative of $x^2+xy+y^3=x$ is $\dfrac{dy}{dx} = \dfrac{1-2x-y}{x+3y^2}$. ∎

► **Example 3.4.3.** Differentiate $\sin(x) = \tan(y)$

Solution. By implicit differentiation,

$$\sin(x) = \tan(y)$$
$$\frac{dy}{dx}\sin(x) = \frac{dy}{dx}\tan(y)$$
$$\cos(x) = \sec^2(y)\cdot\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\cos(x)}{\sec^2(y)}$$

Therefore, the derivative of $\sin(x) = \tan(y)$ is $\dfrac{dy}{dx} = \dfrac{\cos(x)}{\sec^2(y)}$. ∎

► **Example 3.4.4.** Differentiate $\ln(x)+\ln(y) = x+y$

Solution. By implicit differentiation,

$$\frac{d}{dx}\left(\ln(x)+\ln(y)\right) = \frac{d}{dx}\left(x+y\right)$$
$$\frac{d}{dx}\ln(x)+\frac{d}{dx}\ln(y) = \frac{dx}{dx}+\frac{dy}{dx}$$
$$\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx} = 1+\frac{dy}{dx}$$
$$\frac{1}{y}\cdot\frac{dy}{dx}-\frac{dy}{dx} = 1-\frac{1}{x}$$
$$\frac{dy}{dx}\left(\frac{1}{y}-1\right) = 1-\frac{1}{x}$$
$$\frac{dy}{dx} = \frac{1-\frac{1}{x}}{\frac{1}{y}-1}$$

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$$
\begin{aligned}
&= \frac{\dfrac{x-1}{x}}{\dfrac{1-y}{y}} \\
&= \frac{x-1}{x} \cdot \frac{y}{1-y} \\
&= \frac{y(x-1)}{x(1-y)}
\end{aligned}
$$

Therefore, the derivative of $\ln(x) + \ln(y) = x + y$ is $\dfrac{dy}{dx} = \dfrac{y(x-1)}{x(1-y)}$. ∎

► **Example 3.4.5.** Differentiate $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$

► **Example 3.4.6.** Since $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$, therefore, it can be simplified as follows,

$$
\begin{aligned}
y^2 &= \left(\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}\right)^2 \\
y^2 &= x + \underbrace{\sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}_{y} \\
y^2 &= x + y
\end{aligned}
$$

Now, implicitly differentiating the equation above, we get,

$$
\begin{aligned}
\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(x + y) \\
2y\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
2y\frac{dy}{dx} - \frac{dy}{dx} &= 1 \\
\frac{dy}{dx}(2y - 1) &= 1 \\
\frac{dy}{dx} &= \frac{1}{2y - 1} \\
&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} - 1}
\end{aligned}
$$

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Therefore, the derivative of $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$ is

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} - 1}.$$

———————The End of Section 3.4
Implicit Differentiation

§ 3.5 Differentiation Rules for Exponential Functions

Functions of the form $f(x) = a^x$, where a is a constant and x is a variable exponent, are called exponential functions. These functions play a crucial role in describing growth and decay processes for the fact that they grow or shrink at rates proportional to their current value. This is the reason they are ideal for modeling real-world phenomena such as population growth, radioactive decay, and interest calculations. In this section, we will derive the derivatives of exponential functions.

Chap. 3 / Sec. 3.5 / Subsec. 3.5.1 : Differentiation of e^x

In section 3.2, we saw that the derivative of $\sin(x)$ is $\cos(x)$. Similarly, the derivative of $\cos(x)$ is $-\sin(x)$. Now, if $-\sin(x)$ is again differentiated, it turns to $-\cos(x)$. Similarly, if $-\cos(x)$ is again differentiated, the derivative yields $\sin(x)$ and the pattern continues!

Instances such as the one above are indeed interesting, especially, when considering the connection between the rates of change of the function and its derivative. However, what's even more interesting is the existence of a function where the rate at which it changes at a point is equivalent to the value of the function itself at that point!

It may sound unbelievable but it is indicated sufficiently by functions such as $f(x) = 2^x$. For that let us differentiate the function $f(x) = 2^x$ by the definition of a derivative.

$$\begin{aligned}\frac{d}{dx}\left(2^x\right) &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x \cdot 2^h - 2^x}{h}\end{aligned}$$

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$$= \lim_{h\to 0} 2^x \cdot \left(\frac{2^h - 1}{h}\right)$$
$$= 2^x \cdot \lim_{h\to 0} \left(\frac{2^h - 1}{h}\right)$$

Now, we evaluate the limit $\lim_{h\to 0} \left(\frac{2^h - 1}{h}\right)$ numerically by the following table as h approaches to 0,

Table 3.1: Table

h	$\frac{2^h - 1}{h}$
-0.1	0.66967...
-0.01	0.69075...
-0.001	0.69290...
$+0.001$	0.69338...
$+0.01$	0.69555...
$+0.1$	0.71773...

The table above suggests that as $h \to 0$ from both sides of 0, the expression $\frac{2^h - 1}{h} \to k$ such that $0.69290... < k < 0.69338....$ This implies that

$$\frac{d}{dx}(2^x) = k \cdot 2^x$$

This further implies,

$$\frac{d}{dx}(2^x) \propto 2^x$$

This is indeed fascinating that there exists a function $f(x) = 2^x$ where its derivative is proportional to itself. However, this does not answer our search for a function whose derivative is equal to itself. Therefore, let us generalize the case by letting $f(x) = a^x$ this time, where $a \in \mathbb{R}$. If $f(x) = a^x$, then the derivative $\frac{d}{dx}(a^x)$ is,

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h}$$
$$= \lim_{h\to 0} a^x \cdot \left(\frac{a^h - 1}{h}\right)$$
$$= a^x \cdot \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right)$$

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To evaluate the limit $\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)$, we create the following table for different values of a, where $h \in \mathbb{R}$ is a small increment less than 1 and greater than 0.

Table 3.2: Table

a	$\frac{a^h-1}{h}$
2	0.69314...
3	1.09861...
4	1.38630...
5	1.60945...

Now, a clue is found! From the table above, it is visible that between $a = 2$ and $a = 3$, there exists a value of a for which the limit $\lim_{h\to 0}\frac{a^h-1}{h}$ yields 1. And why are we interested $\lim_{h\to 0}\frac{a^h-1}{h}$ resulting in 1? Because then the derivative of a^x (whatever the value of a might be), would be equal to itself. Thus, let us first search the value of a for which $\lim_{h\to 0}\frac{a^h-1}{h} = 1$. To do that, let $\frac{a^h-1}{h} \approx 1$, for a small $h \in \mathbb{R}$. This implies, for a small $h \in \mathbb{R}$,

$$\begin{aligned}\frac{a^h-1}{h} &\approx 1\\ a^h-1 &\approx h\\ a^h &\approx h+1\\ a &\approx \sqrt[h]{1+h}\\ a &\approx (1+h)^{\frac{1}{h}}\end{aligned}$$

We've found the value of a to be approximately equal to $(1+h)^{\frac{1}{h}}$, for a small $h \in \mathbb{R}$. However, for an exact value of a, we let $h \to 0$, that is, the limit,

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = a$$

The numerical value of $(1+h)^{\frac{1}{h}}$ as $h \to 0$, is approximated by the following table,

Table 3.3: Table

h	$(1+h)^{\frac{1}{h}}$
-0.1	$2.86796\cdots$
-0.01	$2.73198\cdots$

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-0.001	$2.71963\cdots$
$+0.001$	$2.71691\cdots$
$+0.01$	$2.70480\cdots$
$+0.1$	$2.59373\cdots$

In general, the numerical value of $(1+h)^{\frac{1}{h}}$ as $h \to 0$ turns out to be $2.718281828459\ldots$. This implies that,

$$\frac{d}{dx}(2.718281828459\ldots)^x = (2.718281828459\ldots)^x$$

Let us denote this real number $2.718281828459\ldots$ as the variable e, that is, $e = 2.718281828459\ldots$, implying that,

$$\frac{d}{dx}(e^x) = e^x$$

As a result, we have found a special function that remains unchanged when differentiated. That is,

Theorem 3.5.1. (The Derivative of the Exponential Function with Base e) If $f(x) = e^x$, where $e \in \mathbb{R}$, then

$$\frac{d}{dx}e^x = e^x$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.5.1.** Differentiate $f(x) = e^{-x}$.

Solution. Since it is known that $\dfrac{d}{dx}e^x = e^x$, therefore only the Chain Rule needs to be further applied in order to differentiate $f(x) = e^{-x}$, as shown below.

$$\begin{aligned}\frac{d}{dx}\left(e^{-x}\right) &= e^{-x}\cdot\frac{d}{dx}\left(-x\right)\\ &= e^{-x}\cdot-\frac{d}{dx}\left(x\right)\\ &= e^{-x}\cdot-1\\ &= -e^{-x}\end{aligned}$$

Thus, the derivative of $f(x) = e^{-x}$ is $f'(x) = -e^{-x}$. ∎

The result of the example above is of such frequency that it should have its own remark, as given below.

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Remark 3.5.1. If $f(x) = e^{-x}$, where $e \in \mathbb{R}$, then

$$\frac{d}{dx}\left(e^{-x}\right) = -e^{-x}$$

► **Example 3.5.2.** Differentiate $f(x) = e^{\sin(x)}$

Solution. The derivative of e^x is itself and the derivative of $\sin(x)$ is $\cos(x)$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left(e^{\sin(x)}\right) &= e^{\sin(x)} \cdot \frac{d}{dx}\Big(\sin(x)\Big) \\ &= e^{\sin(x)}\cos(x)\end{aligned}$$

Thus, the derivative of $f(x) = e^{\sin(x)}$ is $f'(x) = -e^{\sin(x)}\cos(x)$. ∎

► **Example 3.5.3.** Differentiate $f(x) = e^{e^x}$.

Solution. Applying the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\left(e^{e^x}\right) &= e^{e^x} \cdot \frac{d}{dx}\left(e^x\right) \\ &= e^{e^x} \cdot e^x \\ &= e^{e^x + x}\end{aligned}$$

Thus, the derivative of $f(x) = e^{e^x}$ is $f'(x) = e^{e^x+x}$. ∎

Chap. 3 / Sec. 3.5 / Subsec. 3.5.2 : Differentiation of a^x

Now that we have found a function where its derivative matches the function itself, we now try to find the derivative of what was previously unexplored: $f(x) = a^x$. It is known from the discussion above, that

$$\frac{d}{dx}\left(a^x\right) = a^x \cdot \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$$

Now, to solve the limit $\lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$, we use a property of natural logarithms, stating that

$$e^{\ln(x)} = x$$

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Letting $x = a$ yields $e^{\ln(a)} = a$. Therefore, substituting $a = e^{\ln(a)}$ into the limit $\lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$ yields,

$$\begin{aligned} &= \lim_{h\to 0} \frac{a^h - 1}{h} \\ &= \lim_{h\to 0} \frac{\left(e^{\ln(a)}\right)^h - 1}{h} \\ &= \lim_{h\to 0} \frac{e^{\ln(a)h} - 1}{h} \end{aligned} \tag{1}$$

Now, to evaluate the limit in equation (1), we use the following limit,

$$\lim_{h\to 0}\left(\frac{e^h - 1}{h}\right) = 1$$

To apply this limit, let $k = \ln(a)h$. If $k = \ln(a)h$, then $h = \frac{k}{\ln(a)}$. Now, since $h \to 0$, therefore $k \to 0$, and thus,

$$\begin{aligned} &= \lim_{h\to 0} \frac{e^{\ln(a)h} - 1}{h} \\ &= \lim_{k\to 0} \frac{e^k - 1}{\frac{k}{\ln(a)}} \\ &= \underbrace{\lim_{k\to 0} \frac{e^k - 1}{k}}_{\text{the limit is 1}} \cdot \ln(a) \\ &= 1 \cdot \ln(a) \end{aligned}$$

Thus, it can be shown that $\lim_{h\to 0}\left(\frac{a^h - 1}{h}\right) = \ln(a)$ and therefore,

$$\begin{aligned} \frac{d}{dx}\left(a^x\right) &= a^x \cdot \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right) \\ &= a^x \cdot \ln(a) \end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.5.2. (The Derivative of the Exponential Function with Base a) If $f(x) = a^x$, where $a > 0$, then

$$\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$$

A few examples are provided below to demonstrate the theorem above.

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► **Example 3.5.4.** Differentiate $f(x) = 10^x$.

Solution. Letting $a = 10$ yields,

$$\frac{d}{dx}(10^x) = 10^x \cdot \ln(10)$$

Therefore, the derivative of $f(x) = 10^x$ is $f'(x) = 10^x \ln(10)$ ∎

► **Example 3.5.5.** Differentiate $f(x) = (0.36)^x$.

Solution. By the theorem above,

$$\begin{aligned}\frac{d}{dx}2(0.36)^x &= 2 \cdot \frac{d}{dx}(0.36)^x \\ &= 2 \cdot (0.36)^x \cdot \ln(0.36) \\ &= 2\ln(0.36)(0.36)^x\end{aligned}$$

Therefore, the derivative of $f(x) = (0.36)^x$ is $f'(x) = 2\ln(0.36)(0.36)^x$. ∎

► **Example 3.5.6.** Differentiate $f(x) = \ln\left(\pi^{\ln(e)}\right) \cdot \pi^x$.

Solution. By the theorem above,

$$\begin{aligned}\frac{d}{dx}\ln\left(\pi^{\ln(e)}\right) \cdot \pi^x &= \ln\left(\pi^{\ln(e)}\right) \cdot \frac{d}{dx}(\pi^x) \\ &= \ln\left(\pi^1\right) \cdot \pi^x \ln(\pi) \\ &= \pi^x \ln^2(\pi)\end{aligned}$$

Therefore, the derivative of $f(x) = \ln\left(\pi^{\ln(e)}\right) \cdot \pi^x$ is $f'(x) = \pi^x \ln^2(\pi)$. ∎

► **Example 3.5.7.** Differentiate $f(x) = x^a a^x$.

Solution. By the theorem above,

$$\begin{aligned}\frac{d}{dx}(x^a a^x) &= x^a \cdot \frac{d}{dx}(a^x) + a^x \cdot \frac{d}{dx}(x^a) \\ &= x^a \cdot a^x \ln(a) + a^x \cdot a x^{a-1}\end{aligned}$$

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$$= x^a \cdot a^x \ln(a) + a^x \cdot \frac{ax^a}{x}$$
$$= x^a a^x \left(\ln(a) + \frac{a}{x}\right)$$

Therefore, the derivative of $f(x) = x^a a^x$ is $f'(x) = x^a a^x \left(\ln(a) + \frac{a}{x}\right)$. ∎

———————The End of Section 3.5

Differentiation Rules for Exponential Functions

§ 3.6 Differentiation Rules for Logarithmic Functions

Logarithmic functions are the inverse of exponential functions. An exponential function takes the form $y = a^x$, where a is a positive constant, a logarithmic function reverses the process in a way that answers the question—to what exponent a must be raised to obtain y? In this section, we will derive the differentiation rules for the logarithmic functions.

Chap. 3 / Sec. 3.6 / Subsec. 3.6.1 : Differentiation of $\ln(x)$

A natural logarithm $\ln(x)$ is a logarithm with a base e, that is, $\log_e(x) = \ln(x)$. To differentiate natural logarithms, the inverse property of logarithms and exponentials and the Chain Rule are used, along with the implicit differentiation.

We know that for all $x > 0$,

$$x = e^{\ln(x)}$$

Differentiating both sides of the equation with respect to x yields,

$$\frac{d}{dx}(x) = \frac{d}{dx}e^{\ln(x)}$$
$$1 = e^{\ln(x)} \cdot \frac{d}{dx}\ln(x)$$

Now, divide both sides of the equation by $e^{\ln(x)}$. Then

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}$$
$$= \frac{1}{x}$$

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This implies that the derivative of $f(x) = \ln(x)$ is $f'(x) = \dfrac{1}{x}$, for all $x > 0$. Therefore, we have the following theorem.

Theorem 3.6.1. (The Derivative of the Natural Logarithm) If $f(x) = \ln(x)$, where $x > 0$, then

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Note that since the domain of $\ln(x)$ is $(0, \infty)$, therefore $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ is true for all values of $x > 0$. These values can be extended by considering the function $\ln|x|$, instead of $\ln(x)$. Because, by the definition of absolute value, we know,

$$\ln|x| = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

This implies that $\ln|x|$ is defined for all $x \neq 0$. Thus, for $x > 0$, we have

$$\begin{aligned} \frac{d}{dx}\ln|x| &= \frac{d}{dx}\ln(x) \\ &= \frac{1}{x} \end{aligned}$$

And, for $x < 0$, we have,

$$\begin{aligned} \frac{d}{dx}\ln|x| &= \frac{d}{dx}\ln(-x) \\ &= \frac{1}{-x} \cdot \frac{d}{dx}(-x) \\ &= \frac{1}{-x} \cdot -\frac{d}{dx}(x) \\ &= \frac{1}{-x} \cdot -1 \\ &= \frac{1}{x} \end{aligned}$$

Thus, we have the following theorem.

Theorem 3.6.2. (The Derivative of the Natural Logarithm) If $f(x) = \ln|x|$, where $x \neq 0$, then

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Furthermore, if there exists a function $f(x) \neq 0$, then the derivative of $\ln|f(x)|$, happens to be the following.

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)} \cdot \frac{d}{dx}f(x)$$

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$$= \frac{1}{f(x)} \cdot f'(x)$$
$$= \frac{f'(x)}{f(x)}$$

Thus, we have the following theorem.

Theorem 3.6.3. If $\frac{d}{dx}\ln|f(x)|$, where $f(x) \neq 0$, then

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.6.1.** Differentiate $f(x) = x\ln|x|$.

Solution.

$$\begin{aligned}\frac{d}{dx}x\ln|x| &= x \cdot \frac{d}{dx}\ln|x| + \ln|x| \cdot \frac{d}{dx}(x)\\ &= x \cdot \frac{1}{x} + \ln|x| \cdot 1\\ &= 1 + \ln|x|\end{aligned}$$

Therefore, the derivative of $f(x) = x\ln|x|$ is $f'(x) = 1 + \ln|x|$. ∎

► **Example 3.6.2.** Differentiate $f(x) = \ln|\sec(x)|$.

Solution.

$$\begin{aligned}\frac{d}{dx}\ln|\sec(x)| &= \frac{1}{\sec(x)} \cdot \frac{d}{dx}\sec(x)\\ &= \frac{1}{\sec(x)} \cdot \sec(x)\tan(x)\\ &= \tan(x)\end{aligned}$$

Therefore, the derivative of $f(x) = \ln|\sec(x)|$ is $f'(x) = \tan(x)$. ∎

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Chap. 3 / Sec. 3.6 / Subsec. 3.6.2 : Differentiation of $\log_a(x)$

In order to differentiate the common logarithms notice that for all $a > 1$ and $a \neq 0$,

$$y = \log_a(x) \quad \Leftrightarrow \quad x = a^y$$

Now, differentiating both sides of the equation $x = a^y$ with respect to x yields,

$$\frac{d}{dx}(x) = \frac{d}{dx}(a^y)$$
$$1 = a^y \ln(a) \cdot \frac{dy}{dx}$$

Now, divide both sides of the equation by $a^y \ln(a)$. Then

$$\frac{dy}{dx} = \frac{1}{a^y \ln(a)}$$
$$= \frac{1}{x \ln(a)}$$

Therefore, we have the following theorem.

Theorem 3.6.4. (The Derivative of the Common Logarithm) If $f(x) = \log_a(x)$, where $a > 0$ and $a \neq 1$, then

$$\frac{d}{dx}\log_a(x) = \frac{1}{x \ln(a)}$$

For all $x > 0$.

Similar to the natural logarithm, the values of the domain of $\log_a(x)$ can be extended by considering $\log_a|x|$. As a result, we have the following theorem.

Theorem 3.6.5. (The Derivative of the Common Logarithm) If $f(x) = \log_a|x|$, where $a > 0$ and $a \neq 1$, then

$$\frac{d}{dx}\log_a|x| = \frac{1}{x \ln(a)}$$

For all $x \neq 0$.

Similar to the previous subsection, if there exists a function $f(x) \neq 0$, then the derivative of $\log_a|f(x)|$ happens to be the following,

$$\frac{d}{dx}\log_a|f(x)| = \frac{1}{f(x)\ln(a)} \cdot \frac{d}{dx}f(x)$$

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$$= \frac{1}{f(x)\ln(a)} \cdot f'(x)$$
$$= \frac{f'(x)}{f(x)\ln(a)}$$

Thus, we have the following theorem.

Theorem 3.6.6. If $\frac{d}{dx}\log_a |f(x)|$, where $f(x) \neq 0$, then

$$\frac{d}{dx}\log_a |f(x)| = \frac{f'(x)}{f(x)\ln(a)}$$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.6.3.** Differentiate $f(x) = \log_7 (x^7)$.

Solution. By the theorem above,

$$\begin{aligned}\frac{d}{dx}\log_7 (x^7) &= \frac{1}{x^7 \ln(7)} \cdot \frac{d}{dx}(x^7)\\ &= \frac{1}{x^7 \ln(7)} \cdot 7x^6\\ &= \frac{7x^6}{x^7 \ln(7)}\\ &= 7 \cdot \frac{x^6}{x^7} \cdot \frac{1}{\ln(7)}\\ &= 7 \cdot x^{-1} \cdot \frac{1}{\ln(7)}\\ &= \frac{7}{x\ln(7)}\end{aligned}$$

Therefore, the derivative of $f(x) = \log_7 (x^7)$ is $f'(x) = \frac{7}{x\ln(7)}$. ∎

► **Example 3.6.4.** Differentiate $f(x) = \log_\pi(\ln(x))$

Solution. By the theorem above,

$$\begin{aligned}\frac{d}{dx}\log_\pi(\ln(x)) &= \frac{1}{\ln(x)\cdot\ln(\pi)} \cdot \frac{d}{dx}\ln(x)\\ &= \frac{1}{\ln(x)\ln(\pi)} \cdot \frac{1}{x}\end{aligned}$$

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$$= \frac{1}{x \ln(\pi) \ln(x)}$$

Therefore, the derivative of $f(x) = \log_{\pi}(\ln(x))$ is $f'(x) = \dfrac{1}{x \ln(\pi) \ln(x)}$. ∎

THE END OF SECTION 3.6

Differentiation Rules for Logarithmic Functions

§ 3.7 Logarithmic Differentiation

So far, we have discussed several differentiation rules and techniques for different types of functions. However, when a function involves a product or quotient of multiple terms, or when variables appear in complex exponents, the typical differentiation methods can become cumbersome. In such cases, it is often useful to simplify the function before differentiating. A powerful technique for this simplification is to use the properties of logarithms. The basic idea is to apply the natural logarithm to both sides of the equation, simplify the expression using logarithmic properties, and then differentiate implicitly. For example, consider the following function,

$$f(x) = \frac{e^{x+1}x}{x+1}$$

It would be necessary for us to use the the Quotient Rule, the Product Rule, and the chain rule to differentiate the function above. This process would be quite tedious, and simplifying afterward would add additional work and complexity. However, by applying the properties of logarithms, the function can be differentiated with relative ease. But, before that, it is important to review a few properties of natural logarithms, as shown below.

1. $\ln(ab) = \ln(a) + \ln(b)$
2. $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$
3. $\ln\left(a^b\right) = b \ln(a)$
4. $\ln\left(\dfrac{1}{a}\right) = -\ln(a)$

Now, using the properties of natural logarithms shown above, the function can be rewritten as follows:

$$\begin{aligned} \ln f(x) &= \ln\left(\frac{e^{x+1}x}{x+1}\right) \\ &= \ln\left(e^{x+1}x\right) - \ln(x+1) \\ &= \ln\left(e^{x+1}\right) + \ln(x) - \ln(x+1) \end{aligned}$$

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$$
\begin{aligned}
&= (x+1)\ln(e) + \ln(x) - \ln(x+1) \\
&= (x+1) + \ln(x) - \ln(x+1)
\end{aligned}
$$

Now, differentiating both sides of the equation above yields the following,

$$
\begin{aligned}
\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\Big((x+1) + \ln(x) - \ln(x+1)\Big) \\
&= \frac{d}{dx}(x+1) + \frac{d}{dx}\ln(x) - \frac{d}{dx}\ln(x+1) \\
\frac{f'(x)}{f(x)} &= 1 + \frac{1}{x} - \frac{1}{x+1} \\
f'(x) &= \left(1 + \frac{1}{x} - \frac{1}{x+1}\right)\cdot f(x) \\
&= \left(1 + \frac{1}{x} - \frac{1}{x+1}\right)\cdot \frac{e^{x+1}x}{x+1} \\
&= \left(\frac{x(x+1) + (x+1) - x}{x(x+1)}\right)\cdot \frac{e^{x+1}x}{x+1} \\
&= \left(\frac{x^2+x+1}{x(x+1)}\right)\cdot \frac{e^{x+1}x}{x+1} \\
&= \frac{\left(x^2+x+1\right)e^{x+1}}{(x+1)^2} \\
\therefore f'(x) &= \frac{e^{x+1}x^2 + e^{x+1}x + e^{x+1}}{(x+1)^2}
\end{aligned}
$$

Therefore, the derivative $f'(x)$ of the function $f(x) = \dfrac{e^{x+1}x}{x+1}$ is $f'(x) = \dfrac{e^{x+1}x^2 + e^{x+1}x + e^{x+1}}{(x+1)^2}$.

Using the logarithmic differentiation, the proof for the general Power Rule can be given as follows.

Theorem 3.7.1. (The General Power Rule) If $f(x) = x^r$, where $r \in \mathbb{R}$, then

$$\frac{d}{dx}\Big(x^r\Big) = rx^{r-1}$$

Proof. Let $y = x^r$, where $r \in \mathbb{R}$, then,

$$
\begin{aligned}
y &= x^r \\
\ln(y) &= \ln(x^r)
\end{aligned}
$$

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$$\begin{aligned}\ln(y) &= r \cdot \ln(x)\\ \frac{1}{y} \cdot \frac{dy}{dx} &= r \cdot \frac{1}{x}\\ \frac{dy}{dx} &= r \cdot \frac{y}{x}\end{aligned}$$

Since $y = x^r$, therefore,

$$\begin{aligned}\frac{dy}{dx} &= r \cdot \frac{x^r}{x}\\ &= r \cdot x^{r-1}\end{aligned}$$

This completes the proof. ∎

A few examples are provided below to demonstrate the general Power Rule given above.

► **Example 3.7.1.** If $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, then

$$\begin{aligned}\frac{d}{dx}(\sqrt{x}) &= \frac{d}{dx}(x^{\frac{1}{2}})\\ &= \frac{1}{2}x^{\frac{1}{2}-1}\\ &= \frac{1}{2}x^{-\frac{1}{2}}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

Therefore, the derivative of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Remark 3.7.1.

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

► **Example 3.7.2.** If $f(x) = \dfrac{x^{r+1}}{r+1}$, where $r \in \mathbb{R}$, then,

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) &= \frac{1}{r+1} \cdot \frac{d}{dx}\left(x^{r+1}\right)\\ &= \frac{1}{r+1} \cdot \left(r+1 \cdot x^{r+1-1}\right)\\ &= \frac{1}{\cancel{r+1}} \cdot \cancel{r+1} \cdot x^r\end{aligned}$$

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$$= x^r$$

Therefore, the derivative of $f(x) = \dfrac{x^{r+1}}{r+1}$ is $f'(x) = x^r$

Remark 3.7.2.

$$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r$$

where, $r \in \mathbb{R}$.

A few more examples are provided below to demonstrate the logarithmic differentiation.

► **Example 3.7.3.** Differentiate $f(x) = x^x$

Solution. By the properties of logarithms, we write

$$\begin{aligned}\ln f(x) &= \ln\left(x^x\right)\\ &= x \cdot \ln(x)\end{aligned}$$

Differentiating both sides yield,

$$\begin{aligned}\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\left(x \cdot \ln(x)\right)\\ \frac{f'(x)}{f(x)} &= x \cdot \frac{d}{dx}\ln(x) + \ln(x) \cdot \frac{dx}{dx}\\ &= x \cdot \frac{1}{x} + \ln(x) \cdot 1\\ \frac{f'(x)}{f(x)} &= 1 + \ln(x)\\ f'(x) &= (1 + \ln(x))\, f(x)\\ &= (1 + \ln(x))\, x^x\\ &= x^x + x^x \ln(x)\end{aligned}$$

Therefore, the derivative of $f(x) = x^x$ is $f'(x) = x^x + x^x \ln(x)$. ∎

► **Example 3.7.4.** Differentiate $f(x) = \sqrt{x}^{\sqrt{x}}$

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Solution. By the properties of logarithms, we write

$$\begin{aligned}\ln f(x) &= \ln\left(\sqrt{x}^{\sqrt{x}}\right)\\ &= \sqrt{x}\cdot\ln\left(\sqrt{x}\right)\end{aligned}$$

Differentiating both sides yield,

$$\begin{aligned}\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\left(\sqrt{x}\cdot\ln\left(\sqrt{x}\right)\right)\\ \frac{f'(x)}{f(x)} &= \sqrt{x}\cdot\frac{d}{dx}\ln\left(\sqrt{x}\right)+\ln\left(\sqrt{x}\right)\cdot\frac{d}{dx}\sqrt{x}\\ &= \sqrt{x}\cdot\frac{1}{\sqrt{x}}\frac{d}{dx}\sqrt{x}+\ln\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}+\ln\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}\\ \frac{f'(x)}{f(x)} &= \frac{1}{2\sqrt{x}}\left(1+\ln\left(\sqrt{x}\right)\right)\\ f'(x) &= \frac{1+\ln\left(\sqrt{x}\right)}{2\sqrt{x}}\cdot f(x)\\ &= \frac{1+\ln\left(\sqrt{x}\right)}{2\sqrt{x}}\cdot\sqrt{x}^{\sqrt{x}}\\ &= \frac{\sqrt{x}^{\sqrt{x}}+\sqrt{x}^{\sqrt{x}}\ln\left(\sqrt{x}\right)}{2\sqrt{x}}\end{aligned}$$

Therefore, the derivative of $f(x)=\sqrt{x}^{\sqrt{x}}$ is $f'(x)=\dfrac{\sqrt{x}^{\sqrt{x}}+\sqrt{x}^{\sqrt{x}}\ln\left(\sqrt{x}\right)}{2\sqrt{x}}$. ∎

► **Example 3.7.5.** Differentiate $x^y=y^x$

Solution. By the properties of logarithms, we write

$$\begin{aligned}x^y &= y^x\\ \ln\left(x^y\right) &= \ln\left(y^x\right)\\ y\cdot\ln(x) &= x\cdot\ln(y)\end{aligned}$$

Differentiating both sides yield,

$$\begin{aligned}\frac{d}{dx}\left(y\cdot\ln(x)\right) &= \frac{d}{dx}\left(x\cdot\ln(y)\right)\\ y\cdot\frac{d}{dx}\ln(x)+\ln(x)\cdot\frac{dy}{dx} &= x\cdot\frac{d}{dx}\ln(y)+\ln(y)\cdot\frac{dx}{dx}\end{aligned}$$

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$$
\begin{aligned}
y \cdot \frac{1}{x} + \ln(x) \cdot \frac{dy}{dx} &= x \cdot \frac{1}{y}\frac{dy}{dx} + \ln(y) \cdot 1 \\
\frac{y}{x} + \ln(x)\frac{dy}{dx} &= \frac{x}{y}\frac{dy}{dx} + \ln(y) \\
\ln(x)\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} &= \ln(y) - \frac{y}{x} \\
\frac{dy}{dx}\left(\ln(x) - \frac{x}{y}\right) &= \ln(y) - \frac{y}{x} \\
\frac{dy}{dx} &= \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}} \\
&= \frac{\frac{x\ln(y) - y}{x}}{\frac{y\ln(x) - x}{y}} \\
&= \frac{x\ln(y) - y}{x} \cdot \frac{y}{y\ln(x) - x} \\
&= \frac{y\,(x\ln(y) - y)}{x\,(y\ln(x) - x)}
\end{aligned}
$$

Therefore, differentiating $x^y = y^x$, we get $f'(x) = \dfrac{y\,(x\ln(y) - y)}{x\,(y\ln(x) - x)}$. ∎

► **Example 3.7.6.** Differentiate $\ln(xy) = x + y$

Solution. By the properties of logarithms, we write

$$
\begin{aligned}
\ln(xy) &= x + y \\
\ln(x) + \ln(y) &= x + y
\end{aligned}
$$

Differentiating both sides yield,

$$
\begin{aligned}
\frac{d}{dx}\left(\ln(x) + \ln(y)\right) &= \frac{d}{dx}\left(x + y\right) \\
\frac{d}{dx}\ln(x) + \frac{d}{dx}\ln(y) &= \frac{dx}{dx} + \frac{dy}{dx} \\
\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
\frac{1}{y}\frac{dy}{dx} - \frac{dy}{dx} &= 1 - \frac{1}{x} \\
\frac{dy}{dx}\left(\frac{1}{y} - 1\right) &= 1 - \frac{1}{x}
\end{aligned}
$$

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$$\begin{aligned}\frac{dy}{dx} &= \frac{1-\frac{1}{x}}{\frac{1}{y}-1}\\ &= \frac{\frac{x-1}{x}}{\frac{1-y}{y}}\\ &= \frac{x-1}{x}\cdot\frac{y}{1-y}\\ &= \frac{y\,(x-1)}{x\,(1-y)}\end{aligned}$$

Therefore, differentiating $\ln(xy) = x + y$, we get $f'(x) = \dfrac{y\,(x-1)}{x\,(1-y)}$. ∎

———————The End of Section 3.7
Logarithmic Differentiation

§ 3.8 Differentiation Rules for Inverse Trigonometric Functions

Previously, we derived the differentiation rules for the six trigonometric functions. We now derive the differentiation rules for their inverse functions which have various applications across numerous fields.

Chap. 3 / Sec. 3.8 / Subsec. 3.8.1 : Differentiation of $\sin^{-1}(x)$

The derivative of $y = \sin^{-1}(x)$, where $-1 \le x \le 1$, is obtained by implicitly differentiating both sides of $x = \sin(y)$, with respect to x. That is,

$$\begin{aligned}x &= \sin(y)\\ \frac{d}{dx}(x) &= \frac{d}{dx}\sin(y)\\ 1 &= \cos(y)\cdot\frac{dy}{dx}\\ \frac{1}{\cos(y)} &= \frac{\cancel{\cos(y)}\cdot\frac{dy}{dx}}{\cancel{\cos(y)}}\\ \frac{dy}{dx} &= \frac{1}{\cos(y)}\end{aligned} \tag{1}$$

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Now, from the Pythagorean identity, it is known that $\sin^2(\theta) + \cos^2(\theta) = 1$. Letting $\theta = y$ yields,

$$\begin{aligned} \sin^2(y) + \cos^2(y) &= 1 \\ \cos^2(y) &= 1 - \sin^2(y) \\ \cos(y) &= \sqrt{1 - \sin^2(y)} \end{aligned}$$

Now, continuing with equation (1),

$$\begin{aligned} \frac{d}{dx}(y) &= \frac{1}{\cos(y)} \\ &= \frac{1}{\sqrt{1 - \sin^2(y)}} \\ &= \frac{1}{\sqrt{1 - x^2}} \\ \frac{d}{dx}\sin^{-1}(x) &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore, the derivative of $f(x) = \sin^{-1}(x)$ is $f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

Theorem 3.8.1. (The Derivative of the Inverse Sine Function) If $f(x) = \sin^{-1}(x)$, then

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

for $-1 < x < 1$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.1.** Differentiate $f(x) = \left(\sin^{-1}(x)\right)^2$

Solution. Applying the Chain rule, we write,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sin^{-1}(x)\right)^2 \\ &= 2\sin^{-1}(x) \cdot \frac{d}{dx}\sin^{-1}(x) \\ &= 2\sin^{-1}(x) \cdot \frac{1}{\sqrt{1 - x^2}} \\ &= \frac{2\sin^{-1}(x)}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore, the derivative of $f(x) = \left(\sin^{-1}(x)\right)^2$ is $f'(x) = \dfrac{2\sin^{-1}(x)}{\sqrt{1 - x^2}}$ ∎

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► **Example 3.8.2.** Differentiate $f(x) = \dfrac{\sin(x)}{\sin^{-1}(x)}$

Solution. Applying the Quotient Rule, we write,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\sin(x)}{\sin^{-1}(x)}\right) \\
&= \frac{\sin^{-1}(x)\cdot\dfrac{d}{dx}\sin(x) - \sin(x)\cdot\dfrac{d}{dx}\sin^{-1}(x)}{\left(\sin^{-1}(x)\right)^2} \\
&= \frac{\sin^{-1}(x)\cdot\cos(x) - \sin(x)\cdot\dfrac{1}{\sqrt{1-x^2}}}{\left(\sin^{-1}(x)\right)^2} \\
&= \frac{\cos(x)\sin^{-1}(x) - \dfrac{\sin(x)}{\sqrt{1-x^2}}}{\left(\sin^{-1}(x)\right)^2} \\
&= \frac{\dfrac{\sqrt{1-x^2}\cos(x)\sin^{-1}(x)}{\sqrt{1-x^2}} - \dfrac{\sin(x)}{\sqrt{1-x^2}}}{\left(\sin^{-1}(x)\right)^2} \\
&= \frac{\dfrac{\sqrt{1-x^2}\cos(x)\sin^{-1}(x) - \sin(x)}{\sqrt{1-x^2}}}{\left(\sin^{-1}(x)\right)^2} \\
&= \frac{\sqrt{1-x^2}\cos(x)\sin^{-1}(x) - \sin(x)}{\sqrt{1-x^2}\left(\sin^{-1}(x)\right)^2}
\end{aligned}$$

Therefore, $f'(x) = \dfrac{\sqrt{1-x^2}\cos(x)\sin^{-1}(x) - \sin(x)}{\sqrt{1-x^2}\left(\sin^{-1}(x)\right)^2}$ is the derivative of $f(x) = \dfrac{\sin(x)}{\sin^{-1}(x)}$ ∎

Chap. 3 / Sec. 3.8 / Subsec. 3.8.2 : Differentiation of $\cos^{-1}(x)$

The derivative of $y = \cos^{-1}(x)$, where $-1 \le x \le 1$, is obtained by differentiating both sides of $x = \cos(y)$ implicitly with respect to x. That is,

$$\begin{aligned}
x &= \cos(y) \\
\frac{dx}{dx} &= \frac{d}{dx}\cos(y)
\end{aligned}$$

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$$\begin{aligned} 1 &= -\sin(y) \cdot \frac{dy}{dx} \\ \frac{1}{-\sin(y)} &= \frac{\cancel{-\sin(y)} \cdot \frac{dy}{dx}}{\cancel{-\sin(y)}} \\ \frac{dy}{dx} &= \frac{1}{-\sin(y)} \end{aligned} \tag{2}$$

Now, from the Pythagorean identity, it is known that $\sin^2(\theta) + \cos^2(\theta) = 1$. Letting $\theta = y$ yields,

$$\begin{aligned} \sin^2(y) + \cos^2(y) &= 1 \\ \sin^2(y) &= 1 - \cos^2(y) \\ \sin(y) &= \sqrt{1 - \cos^2(y)} \end{aligned}$$

Now, continuing with equation (2),

$$\begin{aligned} \frac{d}{dx}(y) &= -\frac{1}{\sin(y)} \\ &= -\frac{1}{\sqrt{1 - \cos^2(y)}} \\ \frac{d}{dx}\cos^{-1}(x) &= -\frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore, the derivative of $f(x) = \cos^{-1}(x)$ is $f'(x) = -\frac{1}{\sqrt{1 - x^2}}$.

Theorem 3.8.2. (The Derivative of the Cosine Function) If $f(x) = \cos^{-1}(x)$, then

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$$

for $-1 < x < 1$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.3.** Differentiate $f(x) = \cos^{-1}\left(x^2 + 1\right)$

Solution. Applying the Chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\cos^{-1}\left(x^2 + 1\right) \\ &= -\frac{1}{\sqrt{1 - \left(x^2\right)^2}} \cdot \frac{d}{dx}\left(x^2 + 1\right) \end{aligned}$$

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$$= -\frac{1}{\sqrt{1-x^4}} \cdot 2x$$
$$= -\frac{2x}{\sqrt{1-x^4}}$$

Therefore, the derivative of $f(x) = \cos^{-1}\left(x^2+1\right)$ is $f'(x) = -\frac{2x}{\sqrt{1-x^4}}$. ∎

► **Example 3.8.4.** Differentiate $f(x) = \cos^{-1}(\sin(x))$

Solution. By the Chain Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\cos^{-1}(\sin(x)) \\
&= -\frac{1}{\sqrt{1-(\sin(x))^2}} \cdot \frac{d}{dx}\sin(x) \\
&= -\frac{1}{\sqrt{1-(1-\cos^2(x))}} \cdot \cos(x) \\
&= -\frac{\cos(x)}{\sqrt{\cos^2(x)}} \\
&= -\frac{\cancel{\cos(x)}}{\cancel{\cos(x)}} \\
&= -1
\end{aligned}$$

Therefore, the derivative of $f(x) = \cos^{-1}(\sin(x))$ is $f'(x) = -1$. ∎

Chap. 3 / Sec. 3.8 / Subsec. 3.8.3 : Differentiation of $\tan^{-1}(x)$

The derivative of $y = \tan^{-1}(x)$ is obtained by differentiating both sides of $x = \tan(y)$ implicitly with respect to x. That is,

$$\begin{aligned}
x &= \tan(y) \\
\frac{d}{dx}(x) &= \frac{d}{dx}\tan(y) \\
1 &= \sec^2(y) \cdot \frac{dy}{dx} \\
\frac{1}{\sec^2(y)} &= \frac{\cancel{\sec^2(y)} \cdot \frac{dy}{dx}}{\cancel{\sec^2(y)}}
\end{aligned}$$

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$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} \tag{3}$$

Now, from the Pythagorean identity, we know that $\tan^2(\theta) + 1 = \sec^2(\theta)$. Letting $\theta = y$ yields,

$$\sec^2(y) = 1 + \tan^2(y)$$

Now, continuing with equation (3),

$$\begin{aligned}\frac{d}{dx}(y) &= \frac{1}{\sec^2(y)} \\ &= \frac{1}{1+\tan^2(y)} \\ \frac{d}{dx}\tan^{-1}(x) &= \frac{1}{1+x^2}\end{aligned}$$

Therefore, the derivative of $f(x) = \tan^{-1}(x)$ is $f'(x) = \dfrac{1}{1+x^2}$

Theorem 3.8.3. (The Derivative of the Inverse Tangent Function) If $f(x) = \tan^{-1}(x)$, then

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

for $-\infty < x < \infty$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.5.** Differentiate $f(x) = x^2 + \tan^{-1}(x)$

Solution. Applying the elementary rules,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2 + \tan^{-1}(x)\right) \\ &= \frac{d}{dx}x^2 + \frac{d}{dx}\tan^{-1}(x) \\ &= 2x + \frac{1}{1+x^2}\end{aligned}$$

Therefore, the derivative of $f(x) = x^2 + \tan^{-1}(x)$ is $f'(x) = 2x + \dfrac{1}{1+x^2}$ ∎

► **Example 3.8.6.** Differentiate $f(x) = \tan^{-1}(\tan(x))$

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Solution. Applying the Chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\tan^{-1}\left(\tan\left(x\right)\right)\\ &= \frac{1}{1+\tan^2\left(x\right)}\cdot\frac{d}{dx}\tan\left(x\right)\\ &= \frac{1}{1+\tan^2\left(x\right)}\cdot\sec^2\left(x\right)\\ &= \frac{\sec^2\left(x\right)}{1+\tan^2\left(x\right)}\\ &= \frac{\cancel{\sec^2\left(x\right)}}{\cancel{\sec^2\left(x\right)}}\\ &= 1\end{aligned}$$

Therefore, the derivative of $f\left(x\right)=\tan^{-1}\left(\tan\left(x\right)\right)$ is $f'(x)=1$ ∎

Chap. 3 / Sec. 3.8 / Subsec. 3.8.4 : Differentiation of $\cot^{-1}(x)$

The derivative of $y=\cot^{-1}(x)$ is obtained by differentiating both sides of $x=\cot(y)$ implicitly with respect to x. That is,

$$\begin{aligned}\frac{d}{dx}\left(x\right) &= \frac{d}{dx}\cot\left(y\right)\\ 1 &= -\csc^2\left(y\right)\cdot\frac{dy}{dx}\\ \frac{1}{-\csc^2\left(y\right)} &= \frac{\cancel{-\csc^2\left(y\right)}\cdot\frac{dy}{dx}}{\cancel{-\csc^2\left(y\right)}}\\ \frac{dy}{dx} &= -\frac{1}{\csc^2\left(y\right)}\end{aligned}\tag{4}$$

Now, from the Pythagorean identity, we know that $1+\cot^2\left(\theta\right)=\csc^2\left(\theta\right)$. Letting $\theta=y$ yields,

$$\begin{aligned}\csc^2\left(y\right) &= 1+\cot^2\left(y\right)\\ \csc^2\left(y\right) &= 1+x^2\end{aligned}$$

Now, continuing with equation (4),

$$\frac{d}{dx}\left(y\right)=-\frac{1}{\csc^2\left(y\right)}$$

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$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

Therefore, the derivative of $f(x) = \cot^{-1}(x)$ is $f'(x) = -\dfrac{1}{1+x^2}$

Theorem 3.8.4. (The Derivative of the Inverse Cotangent Function) If $f(x) = \cot^{-1}(x)$, then

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

for $-\infty < x < \infty$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.7.** Differentiate $f(x) = \cot^{-1}(e^x)$

Solution. By the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\cot^{-1}(e^x)\\ &= -\frac{1}{1+(e^x)^2}\cdot\frac{d}{dx}e^x\\ &= -\frac{e^x}{1+e^{2x}}\end{aligned}$$

Therefore, the derivative of $f(x) = \cot^{-1}(e^x)$ is $f'(x) = -\dfrac{e^x}{1+e^{2x}}$. ∎

► **Example 3.8.8.** Differentiate $f(x) = \cot(x)\cot^{-1}(x)$

Solution. Applying the Product Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\cot(x)\cot^{-1}(x)\right)\\ &= \cot(x)\cdot\frac{d}{dx}\cot^{-1}(x) + \cot^{-1}(x)\cdot\frac{d}{dx}\cot(x)\\ &= \cot(x)\cdot\left(-\frac{1}{1+x^2}\right) + \cot^{-1}(x)\cdot\left(-\csc^2(x)\right)\\ &= -\frac{\cot(x)}{1+x^2} - \cot^{-1}(x)\csc^2(x)\end{aligned}$$

Therefore, $f'(x) = -\dfrac{\cot(x)}{1+x^2} - \cot^{-1}(x)\csc^2(x)$ is the derivative of $f(x) = \cot(x)\cot^{-1}(x)$ ∎

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Chap. 3 / Sec. 3.8 / Subsec. 3.8.5 : Differentiation of $\sec^{-1}(x)$

The derivative of $y = \sec^{-1}(x)$, is obtained by differentiating both sides of $x = \sec(y)$ implicitly with respect to x, where $x \leq -1$ or $x \geq 1$. That is,

$$\begin{aligned}
\frac{d}{dx}(x) &= \frac{d}{dx}\sec(y) \\
1 &= \sec(y)\cdot\tan(y)\cdot\frac{dy}{dx} \\
\frac{1}{\sec(y)\tan(y)} &= \frac{\cancel{\sec(y)\tan(y)}\cdot\frac{dy}{dx}}{\cancel{\sec(y)\tan(y)}} \\
\frac{dy}{dx} &= \frac{1}{\sec(y)\tan(y)} \qquad (5)
\end{aligned}$$

Now, from the Pythagorean identity, we know that $1 + \tan^2(\theta) = \sec^2(\theta)$. Letting $\theta = y$ yields,

$$\begin{aligned}
1 + \tan^2(y) &= \sec^2(y) \\
\tan^2(y) &= \sec^2(y) - 1 \\
\tan(y) &= \pm\sqrt{\sec^2(y) - 1} \\
\tan(y) &= \pm\sqrt{x^2 - 1}
\end{aligned}$$

continuing with equation (5),

$$\begin{aligned}
\frac{d}{dx}\sec^{-1}(x) &= \frac{1}{\sec(y)\tan(y)} \\
&= \pm\frac{1}{\sec(y)\cdot\sqrt{x^2-1}} \\
&= \pm\frac{1}{x\cdot\sqrt{x^2-1}}
\end{aligned}$$

Now, in order to determine the sign of the derivative above, we need to consider two cases. That is,

1. If $x \geq 1$ in $y = \sec^{-1}(x)$, then $0 \leq y < \frac{\pi}{2}$ and thus $\tan(y) \geq 0$. In this case, $\tan(y) = \sqrt{x^2-1}$
2. If $x \leq 1$ in $y = \sec^{-1}(x)$, then $\frac{\pi}{2} \leq y < \pi$ and thus $\tan(y) \leq 0$. In this case, $\tan(y) = -\sqrt{x^2-1}$

This implies that the derivative of the inverse secant function is,

$$\frac{d}{dx}\sec^{-1}(x) = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1 \end{cases}$$

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The derivative of $y = \sec^{-1}(x)$ above is quite confusing. However, we can simply get around this confusion by using the absolute value function. That is, if we let $|x| > 1$, then we can simplify the derivative of inverse secant as the following.

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

Therefore, the derivative of $f(x) = \sec^{-1}(x)$ is $f'(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, where $|x| > 1$.

Theorem 3.8.5. (The Derivative of the Inverse Secant Function) If $f(x) = \sec^{-1}(x)$, then

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

for $|x| > 1$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.9.** Differentiate $f(x) = \sec(x) + \sec^{-1}(x)$

Solution. By the elementary rules,

$$\begin{aligned}\frac{d}{dx}\left(\sec(x) + \sec^{-1}(x)\right) &= \frac{d}{dx}\sec(x) + \frac{d}{dx}\sec^{-1}(x) \\ &= \sec(x)\tan(x) + \frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

Therefore, the derivative of $f(x) = \sec(x) + \sec^{-1}(x)$ is $f'(x) = \sec(x)\tan(x) + \dfrac{1}{|x|\sqrt{x^2-1}}$ ∎

► **Example 3.8.10.** Differentiate $f(x) = \sec^{-1}(x^e)$

Solution. By the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\sec^{-1}(x^e) &= \frac{1}{|x^e|\sqrt{(x^e)^2-1}} \cdot \frac{d}{dx}x^e \\ &= \frac{1}{|x^e|\sqrt{x^{2e}-1}}ex^{e-1} \\ &= \frac{ex^{e-1}}{\sqrt{(x^e)^2}\sqrt{x^{2e}-1}}\end{aligned}$$

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$$= \frac{e}{x\sqrt{x^{2e}-1}}$$

Therefore, the derivative of $f(x) = \sec^{-1}(x^e)$ is $f'(x) = \dfrac{e}{x\sqrt{x^{2e}-1}}$. ∎

Chap. 3 / Sec. 3.8 / Subsec. 3.8.6 : Differentiation of $\csc^{-1}(x)$

To differentiate $y = \csc^{-1}(x)$, first notice that if $y = \csc^{-1}(x)$, then $x = \csc(y)$, where $x \leq -1$ or $x \geq 1$. Therefore,

$$\begin{aligned}
\csc(y) &= x \\
\frac{1}{\sin(y)} &= x \\
\sin(y) &= \frac{1}{x}
\end{aligned}$$

Now, differentiating both sides of the equation above implicitly, we get,

$$\begin{aligned}
\frac{d}{dx}\sin(y) &= \frac{d}{dx}\left(\frac{1}{x}\right) \\
\cos(y)\cdot\frac{dy}{dx} &= -\frac{1}{x^2} \\
\frac{dy}{dx} &= -\frac{1}{x^2\cos(y)} \qquad (1)
\end{aligned}$$

Now, from the Pythagorean identity, we know that $\sin^2(\theta)+\cos^2(\theta) = 1$. Letting $\theta = y$ and $\sin(y) = \dfrac{1}{x}$ yields,

$$\begin{aligned}
\sin^2(y) + \cos^2(y) &= 1 \\
\left(\frac{1}{x}\right)^2 + \cos^2(y) &= 1 \\
\cos^2(y) &= 1 - \frac{1}{x^2} \\
&= \frac{x^2-1}{x^2} \\
\cos(y) &= \sqrt{\frac{x^2-1}{x^2}} \\
&= \frac{\sqrt{x^2-1}}{\sqrt{x^2}} \\
&= \frac{\sqrt{x^2-1}}{|x|}
\end{aligned}$$

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$$\therefore \cos(y) = \frac{\sqrt{x^2-1}}{|x|}$$

Now, substituting $\cos(y) = \dfrac{\sqrt{x^2-1}}{|x|}$ into equation (1) yields the following,

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{x^2 \cdot \dfrac{\sqrt{x^2-1}}{|x|}} \\ &= -\frac{1}{\dfrac{x^2\sqrt{x^2-1}}{|x|}} \\ &= -\frac{|x|}{x^2\sqrt{x^2-1}}\end{aligned}$$

Since $\dfrac{|x|}{x^2} = \dfrac{1}{|x|}$, therefore,

$$\begin{aligned}\frac{dy}{dx} &= -\frac{|x|}{x^2\sqrt{x^2-1}} \\ &= -\frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

Therefore, the derivative of $f(x) = \csc^{-1}(x)$ is $f'(x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

Theorem 3.8.6. (The Derivative of the Inverse Cosecant Function) If $f(x) = \csc^{-1}(x)$, then

$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

for $|x| > 1$

A few examples are provided below to demonstrate the theorem above.

► **Example 3.8.11.** Differentiate $f(x) = e^x - \csc^{-1}(x)$

Solution. By the elementary rules,

$$\begin{aligned}\frac{d}{dx}\left(e^x - \csc^{-1}(x)\right) &= \frac{d}{dx}e^x - \frac{d}{dx}\csc^{-1}(x) \\ &= e^x + \frac{1}{|x|\sqrt{x^2-1}} \\ &= e^x + \frac{1}{\sqrt{x^2}\sqrt{x^2-1}}\end{aligned}$$

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$$= e^x + \frac{1}{x\sqrt{x^2-1}}$$

Therefore, the derivative of $f(x) = e^x - \csc^{-1}(x)$ is $f'(x) = e^x + \dfrac{1}{x\sqrt{x^2-1}}$. ∎

► **Example 3.8.12.** Differentiate $f(x) = \csc^{-1}(\csc^{-1}(x))$

Solution. By the Chain Rule,

$$\begin{aligned}
\frac{d}{dx}\csc^{-1}(\csc^{-1}(x)) &= -\frac{1}{|\csc^{-1}(x)|\sqrt{(\csc^{-1}(x))^2-1}} \cdot \frac{d}{dx}\csc^{-1}(x) \\
&= -\frac{1}{|\csc^{-1}(x)|\sqrt{(\csc^{-1}(x))^2-1}} \cdot \left(-\frac{1}{|x|\sqrt{x^2-1}}\right) \\
&= \frac{1}{|x|\,|\csc^{-1}(x)|\sqrt{x^2-1}\sqrt{(\csc^{-1}(x))^2-1}} \\
&= \frac{1}{\sqrt{x^2}\sqrt{(\csc^{-1}(x))^2}\sqrt{x^2-1}\sqrt{(\csc^{-1}(x))^2-1}} \\
&= \frac{1}{x\csc^{-1}(x)\sqrt{x^2-1}\sqrt{(\csc^{-1}(x))^2-1}}
\end{aligned}$$

Therefore, the derivative of $f(x) = \csc^{-1}(\csc^{-1}(x))$ is
$f'(x) = \dfrac{1}{x\csc^{-1}(x)\sqrt{x^2-1}\sqrt{(\csc^{-1}(x))^2-1}}$. ∎

THE END OF SECTION 3.8

Differentiation Rules for Inverse Trigonometric Function

§ 3.9 Differentiation of General Inverse Functions

Let $y = f(x)$ be a function with an inverse $x = f^{-1}(y)$ and we want to find the derivative of $x = f^{-1}(y)$ with respect to y. In order to do that, consider the following,

$$\begin{aligned}
y &= f(x) \\
&= f\left(\underbrace{f^{-1}(f(x))}_{x}\right)
\end{aligned}$$

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$$= f\left(f^{-1}(f(x))\right)$$

Now, differentiating both sides of the equation above, by the Chain Rule, with respect to y; we get,

$$\begin{aligned}\frac{dy}{dy} &= \frac{d}{dy}f(f^{-1}(f(x)))\\ 1 &= f'(f^{-1}(f(x))) \cdot \frac{d}{dy}f^{-1}(f(x))\\ \frac{1}{f'(f^{-1}(f(x)))} &= \frac{d}{dy}f^{-1}(f(x))\end{aligned}$$

Since $x = f^{-1}(f(x))$, therefore,

$$\begin{aligned}\frac{d}{dy}f^{-1}(f(x)) &= \frac{1}{f'(f^{-1}(f(x)))}\\ \therefore \frac{dx}{dy} &= \frac{1}{f'(x)}\\ &= \frac{1}{\frac{dy}{dx}}\end{aligned}$$

This implies that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, given that $\frac{dy}{dx} \neq 0$. Therefore, we have the following theorem.

Theorem 3.9.1. (The Derivative of General Inverse Functions) Let $y = f(x)$ a function that is differentiable on an interval I. If $y = f(x)$ has an inverse function $x = f^{-1}(x)$, then the derivative of $x = f^{-1}(x)$ is,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

given that $\frac{dy}{dx} = f'(x) \neq 0$

A few examples to demonstrate the theorem above are provided below.

► **Example 3.9.1.** Differentiate $f(x) = \sin^{-1}(x)$

Solution. Since $y = \sin^{-1}(x)$, therefore $x = \sin(y)$. Therefore,

$$\frac{d}{dy}\sin(y) = \cos(y)$$

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$$\begin{aligned} &= \sqrt{1 - \sin^2(y)} \\ &= \sqrt{1 - x^2} \end{aligned}$$

Since,

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{\frac{dy}{dx}} \\ \sqrt{1 - x^2} &= \frac{1}{\frac{dy}{dx}} \\ \therefore \frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore, $\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$. ∎

► **Example 3.9.2.** Differentiate $f(x) = \cos^{-1}(x)$

Solution. Since $y = \cos^{-1}(x)$, therefore $x = \cos(y)$. Therefore,

$$\begin{aligned} \frac{d}{dy}\cos(y) &= -\sin(y) \\ &= -\sqrt{1 - \cos^2(y)} \\ &= -\sqrt{1 - x^2} \end{aligned}$$

Since,

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{\frac{dy}{dx}} \\ -\sqrt{1 - x^2} &= \frac{1}{\frac{dy}{dx}} \\ \therefore \frac{dy}{dx} &= -\frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Therefore, $\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$. ∎

THE END OF SECTION 3.9

Differentiation of General Inverse Functions

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§ 3.10 Summary

Having covered the derivative of necessary functions, a few of them are summarized for a quick reference. Note that the derivatives summarized below are absolutely crucial to have on your mind for a smooth understanding of the chapters that follow. However, *mindless memorization* is strongly discouraged, even though arguments for being helpful may be made. It is suggested to rather *play*[3] with them until they're unforgettable. Take it as learning to type on a keyboard. While many of us with decent typing speeds may not have memorized each individual location of each key, enough mistakes have been made over time to recall where they are efficiently.

[3]for example, ask why the derivative of $\sin(x)$ cannot be equal to itself, while the derivative of e^x can be; or notice the pattern that the derivative of $\sin^{-1}(x)$ and $\cos^{-1}(x)$, $\tan^{-1}(x)$ and $\cot^{-1}(x)$, $\sec^{-1}(x)$ and $\csc^{-1}(x)$ are almost the same except for a sign in front of them; etc.

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1. $\frac{d}{dx}(c) = 0$
2. $\frac{dx}{dx} = 1$
3. $\frac{d}{dx}x^n = nx^{n-1}$
4. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
5. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
6. $\frac{d}{dx}e^x = e^x$
7. $\frac{d}{dx}e^{-x} = -e^{-x}$
8. $\frac{d}{dx}a^x = a^x \cdot \ln(a)$
9. $\frac{d}{dx}\ln(x) = \frac{d}{dx}\log_e(x) = \frac{1}{x}$
10. $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$
11. $\frac{d}{dx}\sin(x) = \cos(x)$
12. $\frac{d}{dx}\cos(x) = -\sin(x)$
13. $\frac{d}{dx}\tan(x) = \sec^2(x)$
14. $\frac{d}{dx}\cot(x) = -\csc^2(x)$
15. $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
16. $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
17. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
18. $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
20. $\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$
21. $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

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Chapter 4

Applications of Derivatives

In the previous chapters, we discussed the concept of derivatives and their derivation. Now, the focus turns to discussing how these ideas can be applied in practice. The aim is to explore how derivatives can serve as more than just a computational tool and to reveal themselves as the key to solve and understand a wide range of mathematical questions.

§ 4.1 Extreme Values of a Function

Extreme values in mathematics refer to the highest and lowest values that a function can attain, formally termed as *extremum*, indicating either maximum or minimum values of a function. In the plural form, extremum becomes *extrema*, similarly maximum and minimum become *maxima* and *minima*, respectively. These extreme values are categorized into two types: *absolute* and *relative*. An absolute maximum or minimum is the highest or lowest value that a function attains *globally* over a given interval. In contrast, a relative maximum or minimum is the highest or lowest value that a function attains *locally* within a specific neighborhood of points in the interval. Since absolute maximum or minimum values occur globally, therefore absolute maximum or minimum is also said to be the *global* maximum or minimum values. Similarly, since relative maximum or minimum values occur locally, therefore relative maximum or minimum is also said to be the *local* maximum or minimum values. With that in mind, we start our discussion by discussing the absolute extreme values as follows.

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Chap. 4 / Sec. 4.1 / Subsec. 4.1.1 : Absolute (Global) Extreme Values

It may seem intuitive to state that a function reaches its absolute maximum or minimum value, but how can this idea be defined precisely? What does it mean for a function to attain its absolute maximum or minimum values? To define it precisely, consider that if a function attains its absolute maximum value, then all other values of that function must be less than or equal to this maximum, as it is globally the highest value. Similarly, if a function attains its absolute minimum value, then all other values must be greater than or equal to this minimum, as it is globally the lowest value. By this idea, we have the following definition of absolute maximum and minimum values of a function.

Definition 4.1.1. (Absolute Extreme Values) Let $f(x)$ be a function defined on an interval I and let a point $c \in I$, then

1. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is the *absolute maximum* of $f(x)$ over I.
2. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is the *absolute minimum* of $f(x)$ over I.

As suggested by the definition above, the existence of absolute maximum or minimum values of a function depends both on the continuity of the function and the choice of its interval. To illustrate why the choice of the interval matters, consider the continuous function $f(x) = x^2$. Depending on the choice of interval, the existence of absolute maximum or minimum values changes. For example, over the entire domain of $(-\infty, +\infty)$, the function $f(x) = x^2$ has an absolute minimum value at the point $x = 0$, but no absolute maximum value, as shown below.

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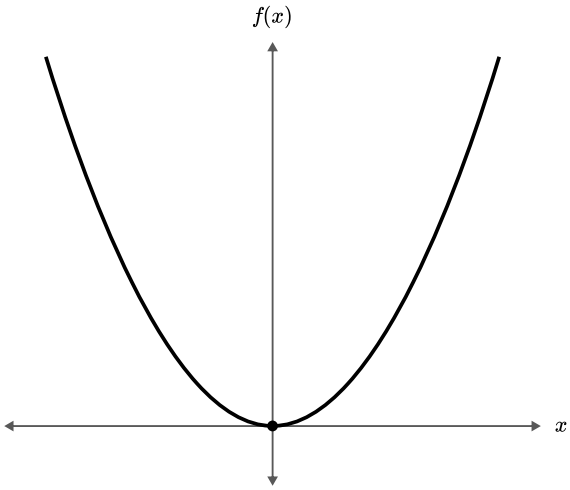

Figure 4.1

However, the same cannot be said for the interval $(0, a]$. Over the interval $(0, a]$, the function $f(x) = x^2$ has an absolute maximum value of $f(a)$ at the point $x = a$, but no absolute minimum value, as shown below.

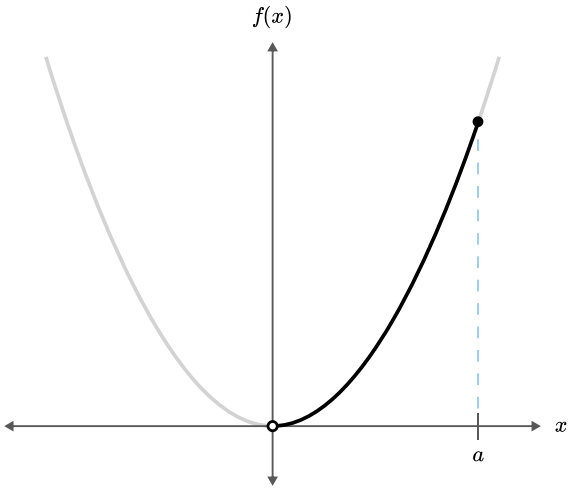

Figure 4.2

Similarly, over the closed interval $[0, a]$, the function $f(x) = x^2$ has both absolute maximum and minimum values of $f(0)$ and $f(a)$ at the points $x = 0$ and $x = a$, respectively. This is shown below.

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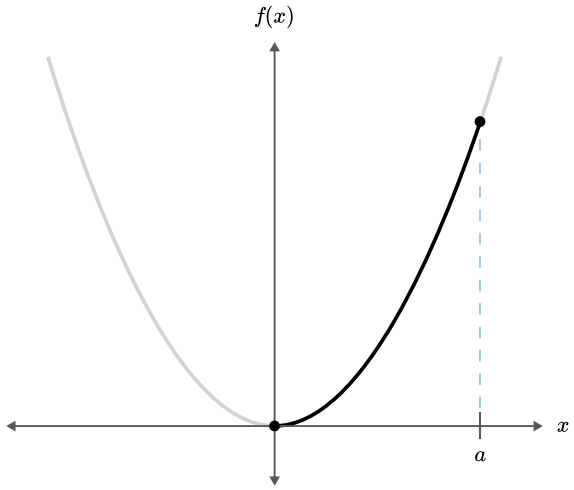

Figure 4.3

Likewise, over the open interval $(0, a)$, the function $f(x) = x^2$ neither has an absolute maximum nor minimum value, as shown below.

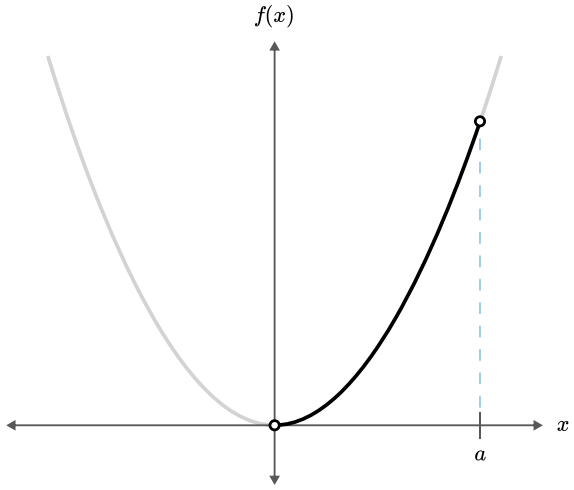

Figure 4.4

This suggests that in order for a function to have any absolute maximum or minimum values over an interval I, the interval I must be bounded, provided that the function is continuous over I. As a result, we have the following theorem.

Theorem 4.1.1. (Extreme Value Theorem) Let $f(x)$ be a function satisfying the following conditions.

1. $f(x)$ is continuous over an interval I.
2. The interval I is closed.

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Then, there exists an absolute maximum and minimum value of the function $f(x)$ over the interval I.

The theorem above is the *existence theorem*, which provides conditions for absolute maximum or minimum values of a function. However, it does not provide a method for finding these values. To determine how to find them, it is necessary to understand the concept of "critical points," which in turn requires knowledge of the *relative* extreme values of a function. As a result, we now discuss the relative extreme values.

Chap. 4 / Sec. 4.1 / Subsec. 4.1.2 : Relative (Local) Extreme Values

To understand the concept of Relative or Local Extreme (maximum or minimum) values, consider the figure of a continuous function below.

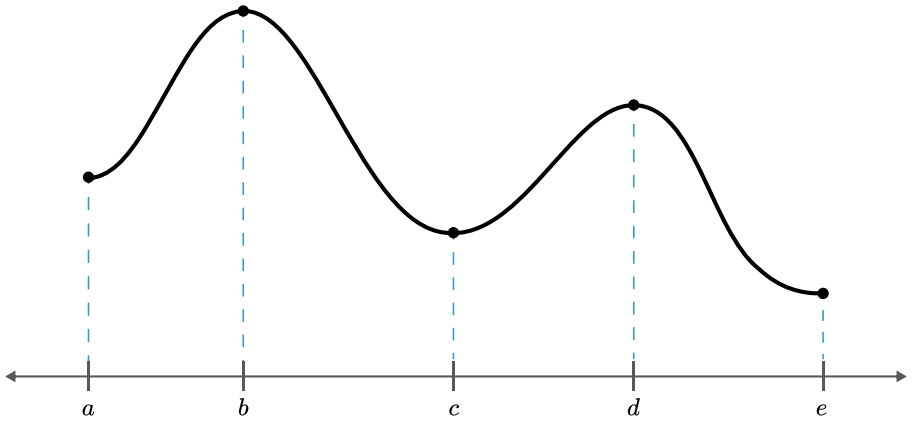

Figure 4.5

At which points does the function attain its absolute maximum and minimum values? Clearly, the function attains its absolute maximum at $x = b$ and its absolute minimum at $x = e$. However, at the points c and d, the function also shows some characteristics of being a maximum or minimum. Specifically, at these points, the function is a maximum or minimum *relative to the neighboring points*. Thus, at the point d, the function has a *relative* maximum value. Similarly, at the point c, the function has a *relative* minimum value.

Note that relative maximum or minimum values of a function occur at interior points of an interval rather than at its endpoints. Therefore, we have the following definition of relative extreme values.

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Definition 4.1.2. (Relative Extreme Values) Let $f(x)$ be a function defined on an interval I and let $c \in I$, then,

1. If $f(c) \geq f(x)$ for all x near c, then $f(c)$ is the *relative maximum* of $f(x)$ over I.
2. If $f(c) \leq f(x)$ for all x near c, then $f(c)$ is the *relative minimum* of $f(x)$ over I.

Now that we have discussed the relative extreme values, let's now move on to critical points. These points are essential for finding the absolute extreme values of a function.

Chap. 4 / Sec. 4.1 / Subsec. 4.1.3 : Critical Points

To understand the concept of critical points, consider again the graph of a continuous function as shown below.

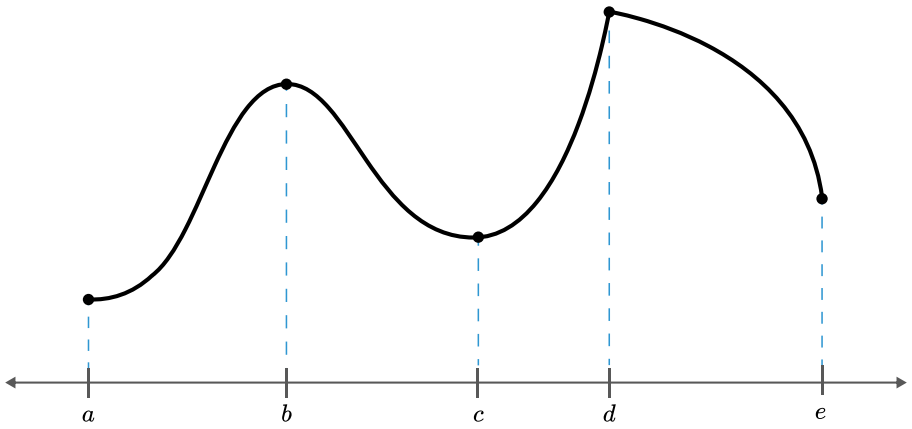

Figure 4.6

At which points does the function attain its relative maximum and minimum values? As before, it is visible that the function has a relative maximum value at the points $x = b$ and $x = d$. Likewise, it has a relative minimum value at the points $x = c$. Now, notice how the function only has its relative maximum and minimum values at points where the derivative is either zero (at the point $x = b$) or fails to exist (at the point $x = d$).

Based on that observation, we have the following theorem.

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Theorem 4.1.2. (Relative/Local Extreme Value Theorem)
If $f(x)$ has a relative maximum or minimum value at the point c and is differentiable at c, then

$$f'(c) = 0$$

Proof. To prove that $f'(c) = 0$ at a relative extremum, we show that neither $f'(c)$ cannot be solely positive nor be solely negative at a relative extremum, and thus $f'(c)$ must be equal to 0. For that, consider the definition of the derivative of a function,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

We assume that the limit exists, and therefore the right-side and the left-side limits also exist at $x = c$ and are equal to $f'(c)$. Now, separating these limits yields the following.

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - a} \quad \text{and} \quad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

Now, we consider the following cases.

Case 1. $f(x)$ has a relative maximum at the point $x = c$.

This implies, from the definition of relative maximum, that for x near c, $f(c) \geq f(x)$ or $f(x) - f(c) \leq 0$. Now,

As $x \to c^+$: As $x \to c^+$, then $x > c$ for all x near c. This implies that $x - c > 0$. Therefore,

$$\frac{\overbrace{f(x) - f(c)}^{\text{negative or zero}}}{\underbrace{x - c}_{\text{positive}}} \leq 0$$

$$\lim_{x \to c^-} \frac{\overbrace{f(x) - f(c)}^{\text{negative or zero}}}{\underbrace{x - c}_{\text{positive}}} \leq 0$$

As $x \to c^-$: As $x \to c^-$, then $x < c$ for all x near c. This implies that $x - c < 0$.

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Therefore,

$$\frac{\overbrace{f(x)-f(c)}^{\text{negative or zero}}}{\underbrace{x-c}_{\text{negative}}} \geq 0$$

$$\lim_{x\to c^-} \frac{\overbrace{f(x)-f(c)}^{\text{negative or zero}}}{\underbrace{x-c}_{\text{negative}}} \geq 0$$

This implies that the limit $f'(c) = \lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ must be 0.

Case 2. $f(x)$ has a relative minimum at the point $x = c$.

This implies, from the definition of relative minimum, that for x near c, $f(c) \leq f(x)$ or $0 \leq f(x) - f(c)$. Now,

As $x \to c^+$: As $x \to c^+$, then $x > c$ for all x near c. This implies that $x - c > 0$. Therefore,

$$\frac{\overbrace{f(x)-f(c)}^{\text{positive or zero}}}{\underbrace{x-c}_{\text{positive}}} \geq 0$$

$$\lim_{x\to c^-} \frac{\overbrace{f(x)-f(c)}^{\text{positive or zero}}}{\underbrace{x-c}_{\text{positive}}} \geq 0$$

As $x \to c^-$: As $x \to c^-$, then $x < c$ for all x near c. This implies that $x - c < 0$. Therefore,

$$\frac{\overbrace{f(x)-f(c)}^{\text{positive or zero}}}{\underbrace{x-c}_{\text{negative}}} \leq 0$$

$$\lim_{x\to c^-} \frac{\overbrace{f(x)-f(c)}^{\text{positive or zero}}}{\underbrace{x-c}_{\text{negative}}} \leq 0$$

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This again implies that the limit $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ must be 0. This completes the proof.

∎

The relative maximum or minimum values of a function can occur not only at points where $f'(c) = 0$, but also at points where the derivative does not exist. Taking these conditions into account, we define such points as "Critical Points."

Definition 4.1.3. (Critical Point) Let $f(x)$ be a function defined over an interval $[a, b]$ and a point $c \in (a, b)$. Now, c is said to be a *Critical Point* of $f(x)$, if either one of the following condition is satisfied:

1. $f'(c) = 0$
2. $f'(c) =$ undefined

The definition above implies that if a function has a relative maximum or minimum value at a point c, then c must be a Critical Point. However, a Critical Point at c does not guarantee a relative maximum or minimum value at that point.

This is because it is possible that $f'(c) = 0$, but there are no relative maximum or minimum values of the function at that point, as shown below.

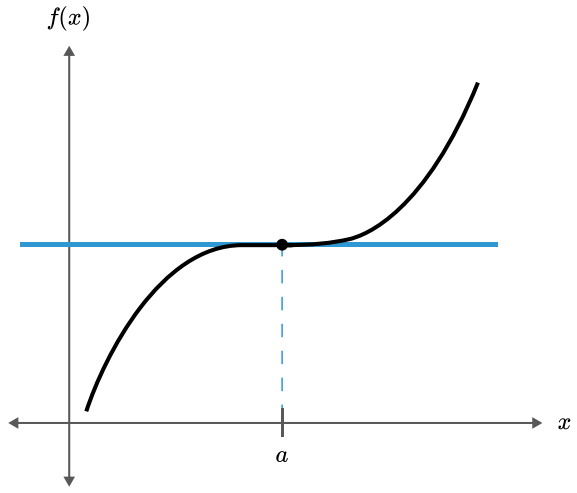

Figure 4.7

Similarly, it is possible that $f'(c) =$ undefined, yet there are no relative maximum or minimum values of the function at that point, as shown below.

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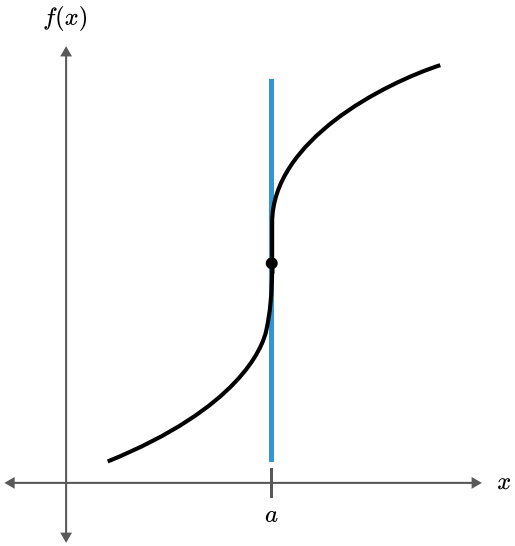

Figure 4.8

This observation indicates that Critical Points are not necessarily where the relative maximum or minimum values of a function occur; rather, they are *potential locations*—a *candidate*—where these extrema might be found.

► **Example 4.1.1.** Find the Critical Points of $f(x) = 10x^2 - 5x^2$

Solution. The derivative of $f(x) = 10x^2 - 5x^2$ is,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(10x^2 - 5x^2\right) \\ &= 20x - 5\end{aligned}$$

Since $f'(x) = 20x - 5$ is a linear function, therefore it is defined for all $x \in \mathbb{R}$. Therefore, to find the Critical Points, we set $f'(x) = 0$, as follows.

$$\begin{aligned}f'(x) &= 0 \\ 20x - 5 &= 0 \\ x &= \frac{5}{20}\end{aligned}$$

Therefore, the only Critical Point of the function $f(x) = 10x^2 - 5x^2$ is $x = \frac{5}{20}$ ∎

► **Example 4.1.2.** Find the Critical Points of $f(x) = x^3 - x$

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Solution. The derivative of $f(x) = x^3 - x$ is,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 - x\right)\\ &= 3x^2 - 1\end{aligned}$$

Since $f'(x) = 3x^2 - 1$ is a quadratic function, therefore it is defined for all $x \in \mathbb{R}$. Therefore, to find the Critical Points, set $f'(x) = 0$, as follows.

$$\begin{aligned}f'(x) &= 0\\ 3x^2 - 1 &= 0\\ x &= \pm\sqrt{\frac{1}{3}}\end{aligned}$$

Therefore, the Critical Points of the function $f(x) = x^3 - x$ are $x = +\sqrt{\frac{1}{3}}$ and $x = \pm - \sqrt{\frac{1}{3}}$, respectively. ∎

Having discussed Critical Points and their identification, we now return to the topic that was left unexplored: locating absolute extreme values.

Chap. 4 / Sec. 4.1 / Subsec. 4.1.4 : Location of Absolute (Global) Extreme Values

The Extreme Value Theorem guarantees the existence of absolute maximum or minimum values under certain conditions. However, it does not provide any method to determine them. Nevertheless, with an understanding of relative extreme values, the following assumptions can be made.

1. Absolute extreme values can occur at the endpoints of a closed interval.
2. Absolute extreme values can occur at the relative extreme values.

Recall that the relative extreme values occur at the Critical Points. Thus, with the knowledge of locating such Critical Points, the absolute extreme values can now be located with almost no effort.

As a result, the assumption above can be turned into a general guideline for locating absolute extreme values as follows.

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Let $f(x)$ be a function defined over a closed interval $[a, b]$. Then, to locate the absolute maximum or minimum values of $f(x)$, consider the following steps:

1. Find the Critical Points of $f(x)$
2. Evaluate $f(x)$ for each Critical Point found in Step 01.
3. Evaluate $f(x)$ for the endpoints of $[a, b]$
4. The greatest of the evaluated values is the maximum and the smallest of the evaluated values is the minimum value of $f(x)$.

A few examples are presented below to demonstrate the guidelines above.

► **Example 4.1.3.** Find the absolute maximum and minimum values of the function $f(x) = x^4 - 2x$ over $[-2, 2]$

Solution. In order to find the absolute maximum and minimum values of the function $f(x) = x^4 - 2x$ over $[-2, 2]$, the Critical Points must first be determined. In order to determine the Critical Points, the function $f(x) = x^4 - 2x$ needs to be differentiated as follows.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^4 - 2x\right) \\ &= 4x^3 - 2 \end{aligned}$$

Now, the derivative $f'(x) = 4x^3 - 2$ is defined for all $x \in \mathbb{R}$. Therefore, to find the Critical Points, let $f'(x) = 0$ as follows,

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 2 &= 0 \\ x &= \sqrt[3]{\frac{1}{2}} \end{aligned}$$

Therefore, the Critical Point of the function $f(x) = x^4 - 2x$ is $x = \sqrt[3]{\frac{1}{2}}$.

This Critical Point, along with the endpoints $[-2, 2]$ are the possible points where the function has its absolute maximum and minimum values. Therefore, solving the function at those points yields,

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Left-endpoint: $f(-2) = (-2)^4 - 2(-2) = 20$

Critical Point: $f\left(\sqrt[3]{\frac{1}{2}}\right) = \left(\sqrt[3]{\frac{1}{2}}\right)^4 - 2\left(\sqrt[3]{\frac{1}{2}}\right) \approx -1.191$

Right-endpoint: $f(2) = (2)^4 - 2(2) = 12$

Comparing these values, it is evident that $-1.191 < 12 < 20$. Thus, the function attains its maximum value at $x = 20$ and minimum value at $x \approx -1.191$. ▮

► **Example 4.1.4.** Find the absolute maximum and minimum values of the function $f(x) = 2x^{\frac{2}{3}} - 4x$ over $[-1, 1]$

Solution. First, differentiate $f(x)$ to find the Critical Points, as follows.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^{\frac{2}{3}} - 4x\right) \\ &= 2 \cdot \frac{2}{3}x^{-\frac{1}{3}} - 4 \\ &= \frac{4}{3} \cdot \frac{1}{x^{\frac{1}{3}}} - 4 \\ &= \frac{4}{3x^{\frac{1}{3}}} - 4 \\ &= 4\left(\frac{1 - 3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}\right) \end{aligned}$$

Now, when $x = 0.037$, the function $f(x)$ has a Critical Point. Because at $x = 0.037$ the derivative of $f(x)$ yields $f'(0.037) = 0$. Similarly, when $x = 0$, the function $f(x)$ has a Critical Point. Because at $x = 0$, the derivative of $f(x)$ yields $f'(0) =$ division by zero $=$ undefined. Therefore, the Critical Points of the function $f(x) = 2x^{\frac{2}{3}} - 4x$ are $x = 0, 0.037$.

The Critical Points along with the endpoints $[-1, 1]$ are the possible points for maximum and minimum values. Therefore, solving the function at those points yields,

Left-endpoint: $f(-1) = 2(-1)^{\frac{2}{3}} - 4(-1) = 6$

Critical Point 1: $f(0) = 2(0)^{\frac{2}{3}} - 4(0) = 0$

Critical Point 2: $f(0.037) = 2(0.037)^{\frac{2}{3}} - 4(0.037) = 0.074$

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Right-endpoint: $f(1) = 2(1)^{\frac{2}{3}} - 4\,(1) = -2$

By comparing these values, it is evident that $-2 < 0 < 0.074 < 6$. Therefore, the function attains its maximum value at $x = -1$ and minimum value at $x = 1$. ∎

———————The End of Section 4.1

Extreme Values of a Function

§ 4.2 Increasing and Decreasing Functions

It is intuitively clear from the graph of a function whether it is increasing, decreasing, or constant. For example, consider the graph of a function below.

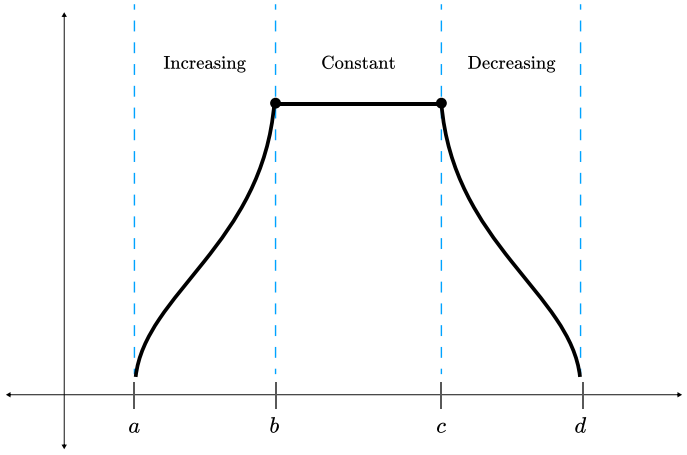

Figure 4.9

In the figure above, the function is increasing over the interval $[a, b]$, constant over $[b, c]$, and then decreasing over $[c, d]$. That is, when x increases over the interval $[a, b]$, the function moves up. Over the interval $[b, c]$, the function remains the same. And lastly, it moves down, as x increases over the interval $[c, d]$.

This intuitive understanding, however, is not mathematically sufficient for a number of reasons and can be made mathematically rigorous as discussed below.

Chap. 4 / Sec. 4.2 / Subsec. 4.2.1 : Increasing and Decreasing Functions

To rigorously define an increasing and decreasing function, consider any two points $x_1, x_2 \in [a, b]$ where $x_1 < x_2$. If a function is increasing over an interval,

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then $f(x_1) < f(x_2)$, as shown in Figure 4.10(a). In contrast, if a function is decreasing over an interval, then $f(x_1) > f(x_2)$, as shown in Figure 4.10(b). And, if a function is constant over an interval, then $f(x_1) = f(x_2)$, as shown in Figure 4.10(c).

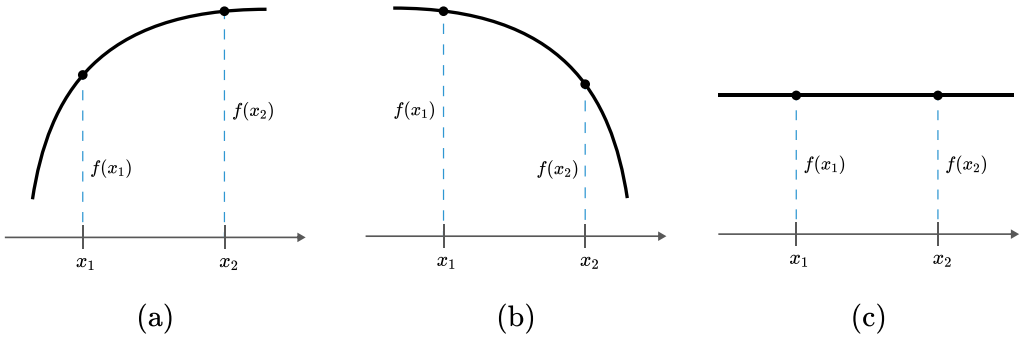

Figure 4.10

Summarizing the discussion, we have the following definition of an increasing, decreasing, and constant function.

Definition 4.2.1. (Increasing, Decreasing, and Constant Functions)
Let $f(x)$ be a function defined over an interval I and let two points $x_1, x_2 \in I$. Then

1. If $f(x_1) < f(x_2)$ for $x_1 < x_2$, then $f(x)$ is *increasing* over I.
2. If $f(x_1) > f(x_2)$ for $x_1 < x_2$, then $f(x)$ is *decreasing* over I.
3. If $f(x_1) = f(x_2)$ for $x_1 < x_2$, then $f(x)$ is *constant* over I.

Since the precise definitions of increasing, decreasing, and constant functions have been established, we now shift our attention to identify intervals where the function shows such behaviors. This identification is done through the use of the derivative, as it gives the slopes of tangent lines. When the derivative is positive on an interval, the tangent lines on that interval have positive slopes. Therefore, the function is increasing on that interval, as shown below.

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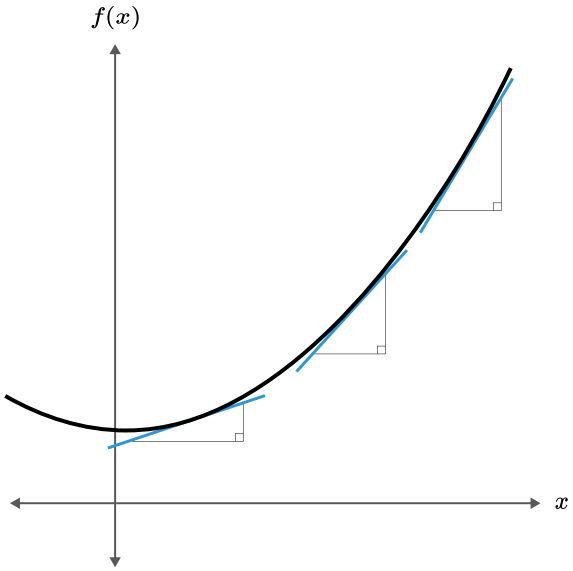

Figure 4.11

Similarly, when the derivative is negative on an interval, the tangent lines on that interval have negative slopes. Therefore, the function is decreasing on that interval, as shown below.

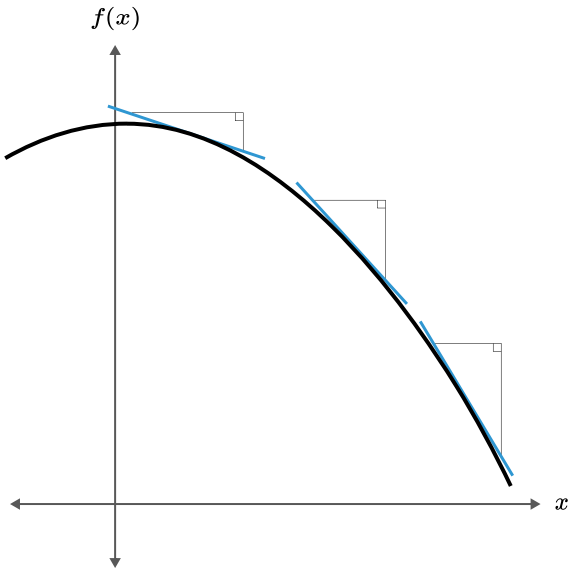

Figure 4.12

Likewise, when the derivative is zero, the tangent line on that interval represents a horizontal line, suggesting the function is constant on that interval.

With this intuition, we have the following theorem to identify intervals on which a function is increasing, decreasing, or constant.

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Theorem 4.2.1. (Increasing and Decreasing Test)
Let $f(x)$ be a continuous function over the interval $[a, b]$ and differentiable over the interval (a, b). Then,

1. If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is *increasing* over $[a, b]$.
2. If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is *decreasing* over $[a, b]$.
3. If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is *constant* over $[a, b]$.

► **Example 4.2.1.** Find the intervals on which the function $f(x) = \frac{10}{3}x^3 - 10x$ is increasing and decreasing.

Solution. It is known from the theorem above that a function $f(x)$ is increasing if its derivative $f'(x)$ is greater than 0, and decreasing if its derivative $f'(x)$ is less than 0. Therefore, the first task would be to differentiate the function $f(x) = \frac{10}{3}x^3 - 10x$, as follows.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{10}{3}x^3 - 10x\right) \\ &= \frac{d}{dx}\left(\frac{10}{3}x^3\right) - \frac{d}{dx}(10x) \\ &= \frac{10}{\cancel{3}} \cdot \cancel{3}x^2 - 10 \\ &= 10x^2 - 10 \\ &= 10\left(x^2 - 1\right) \end{aligned}$$

Now, since $f(x)$ is increasing if $f'(x) = 10\left(x^2 - 1\right) > 0$ and decreasing if $f'(x) = 10\left(x^2 - 1\right) < 0$, therefore, we first need to evaluate $f'(x) = 0$ and the result would be the Critical Points of $f(x)$.

$$\begin{aligned} f'(x) &= 0 \\ 10\left(x^2 - 1\right) &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm\sqrt{1} \\ x &= \pm 1 \end{aligned}$$

The Critical Points are $x = 1$ and $x = -1$. The Critical Points divide the real line into the following intervals:

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$$(-\infty, -1) \qquad (-1, 1) \qquad (1, \infty)$$

Now, for the first interval, choose a value of x such that it lies within that interval. Say, $x = -2 \in (-\infty, -1)$ is chosen as a test value. Now, evaluate $f'(x) = 10\left(x^2 - 1\right)$ for $x = -2$. If the result yields a positive value, then $f(x)$ is increasing within that interval. So,

$$\begin{aligned} f'(-2) &= 10\left((-2)^2 - 1\right) \\ &= 30 > 0 \end{aligned}$$

Since $f'(x) > 0$ over the interval $(-\infty, -1)$, therefore $f(x)$ is increasing over $(-\infty, -1)$.

Similarly, let $x = 0 \in (-1, 1)$ as a test value for the second interval. Then,

$$\begin{aligned} f'(0) &= 10\left(0^2 - 1\right) \\ &= -10 < 0 \end{aligned}$$

Since $f'(x) < 0$ over the interval $(-1, 1)$, therefore $f(x)$ is decreasing over $(-1, 1)$.

Likewise, let $x = 2 \in (1, \infty)$ as a test value for the third interval. Then,

$$\begin{aligned} f'(2) &= 10\left(2^2 - 1\right) \\ &= 30 > 0 \end{aligned}$$

Since $f'(x) > 0$ over the interval $(1, \infty)$, therefore $f(x)$ is increasing over $(1, \infty)$.

In conclusion, the function $f(x)$ is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 1)$. ∎

As previously discussed, if a function $f(x)$ has a relative (local) maximum or minimum value at c, then c must be a Critical Point of $f(x)$. However, not all Critical Points correspond to relative maximum or minimum values. As a result, it is necessary to have a method—referred to as the "First Derivative Test"—to determine whether a function $f(x)$ has a relative maximum or minimum at a given Critical Point. In order to do so, we need to have a firm understanding of the concept of increasing/decreasing functions as already discussed above. Therefore, let us now begin our discussion of the First Derivative Test as follows.

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Chap. 4 / Sec. 4.2 / Subsec. 4.2.2 : The First Derivative Test

With an understanding of the concepts of increasing, decreasing, and constant functions, as well as the methods to determine such, it is now possible to apply this knowledge to identify local extreme values of a function. In order to do that, consider the following graph of a function,

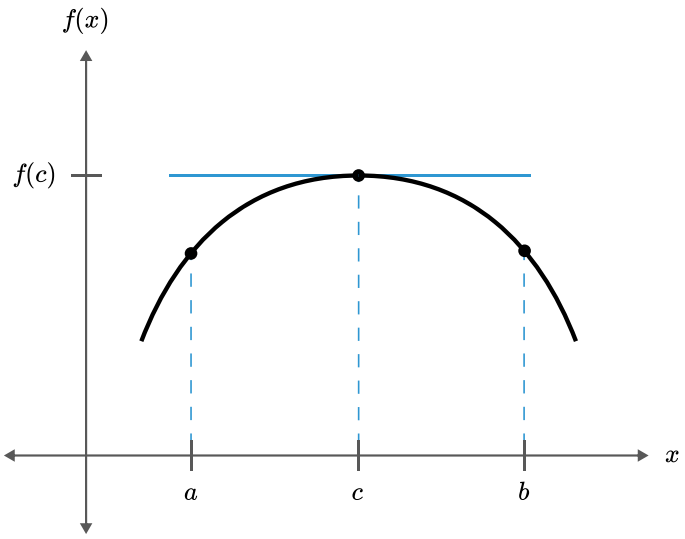

Figure 4.13

In the graph above, the function $f(x)$ has a Critical Point at $x = c$, as $f'(c) = 0$. Now, observe that $f(x)$ is increasing over the interval (a, c) and decreasing over the interval (c, b). Which suggests that $f'(x) > 0$ over the interval (a, c) and $f'(x) < 0$ over (c, b). The derivative $f'(x)$ changes its sign from positive to negative at the Critical Point $x = c$, where $f'(c) = 0$. As a result, the function $f'(x)$ has a relative maximum at the Critical Point $x = c$.

Similarly, for the relative minimum values, consider the following graph of a function.

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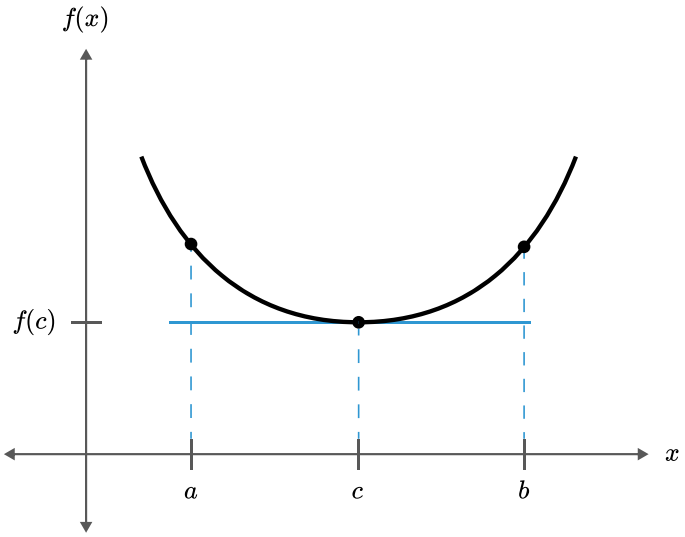

Figure 4.14

Notice that $f(x)$ is decreasing over the interval (a, c) and increasing over the interval (c, b). This suggests that $f'(x) < 0$ over the interval (a, c) and $f'(x) > 0$ over (c, b). The derivative $f'(x)$ changes its sign from negative to positive at the Critical Point $x = c$, where $f'(c) = 0$. As a result, the function $f'(x)$ has a relative minimum at the Critical Point $x = c$.

From the observation above, it can be noted that when a function $f(x)$ has a relative maximum or minimum value at a point, the derivative $f'(x)$ changes its sign at that point. This idea is crucial for identifying the relative maximum or minimum values of a function.

Recall that Critical Points are points where a function *might* have a relative maximum or minimum, but is not guaranteed. The derivative $f'(x)$ does not always change sign at these points. Therefore, examining whether the derivative changes sign at Critical Points can serve as a test to identify if they correspond to any relative maximum or minimum values of the function. Thus, we have the following theorem to identify relative extreme values of a function.

Theorem 4.2.2. (The First Derivative Test)
Let $f(x)$ be a continuous function over an interval I containing a Critical Point $x = c$. If $f(x)$ is differentiable over that interval (except possibly at point c), then

1. If the sign of $f'(x)$ changes *from positive* $(-)$ *to negative* $(+)$ at c,

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then $f(c)$ is a *relative maximum.*

2. If the sign of $f'(x)$ changes *from negative* $(+)$ *to positive* $(-)$ at c, then $f(c)$ is a *relative minimum*

3. If the sign of $f'(x)$ remains the *same* from both sides of c, then $f(c)$ is *neither a relative maximum nor a relative minimum.*

A few examples to demonstrate the theorem above are shown below.

► **Example 4.2.2.** Identify the relative maximum and minimum values of the function $f(x) = \frac{10}{3}x^3 - 10x$.

Solution. To identify the relative extreme values of the function, we need to know the intervals on which the function is increasing and decreasing. From Example 4.2.1, it was known that the function $f(x) = \frac{10}{3}x^3 - 10x$ is increasing on $(-\infty, -1)$, decreasing on $(-1, 1)$, and then again increasing on $(1, \infty)$.

In Example 4.2.1, it has been observed through the test points that $f'(x)$ changes its sign from positive to negative as x goes through the Critical Point $x = -1$. Thus, $f(x)$ has a relative maximum value at $x = -1$. Similarly, since $f'(x)$ changes its sign from negative to positive as x goes through the Critical Point $x = 1$, therefore $f(x)$ has a relative minimum value at $x = 1$. ∎

► **Example 4.2.3.** Identify the relative maximum and minimum values of the function $f(x) = \dfrac{\left(x^2 + 4x\right)\left(x^2 + 4x + 6\right)}{4}$

Solution. To identify the relative extreme values of the function, the first step is to find its derivative to locate the Critical Points. Then, by analyzing whether the derivative changes sign at these Critical Points, the relative extreme values can be determined. Thus, the derivative of $f(x)$ is as follows,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{\left(x^2 + 4x\right)\left(x^2 + 4x + 6\right)}{4}\right) \\ &= \frac{1}{4} \cdot \left(\left(x^2 + 4x\right) \cdot \frac{d}{dx}\left(x^2 + 4x + 6\right) + \left(x^2 + 4x + 6\right) \cdot \frac{d}{dx}\left(x^2 + 4x\right)\right) \\ &= \frac{1}{4} \cdot \left(\left(x^2 + 4x\right) \cdot (2x + 4) + \left(x^2 + 4x + 6\right) \cdot (2x + 4)\right) \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{4} \cdot \left(\left(2x^3 + 12x^2 + 16x\right) + \left(2x^3 + 12x^2 + 28x + 24\right)\right) \\
&= \frac{1}{4} \cdot \left(4x^3 + 24x^2 + 44x + 24\right) \\
&= x^3 + 6x^2 + 11x + 6 \\
&= (x+1)\,(x+2)\,(x+3)
\end{aligned}
$$

Now, since $f(x)$ is continuous at every point, therefore the $f'(x) = 0$ are the only Critical Points. Letting $f'(x) = 0$ yields,

$$
\begin{aligned}
(x+1)\,(x+2)\,(x+3) &= 0 \\
x &= -1, -2, -3
\end{aligned}
$$

These Critical Points divide the real number line into the following intervals.

$$(-\infty, -3) \quad (-3, -2) \quad (-2, -1) \quad (-1, \infty)$$

Now, based on these intervals, we analyze the function to check if it changes its sign and then conclude whether it is increasing or decreasing to identify the relative extreme values, as follows.

Intervals	$(-\infty, -3)$	$(-3, -2)$	$(-2, -1)$	$(-1, \infty)$
Test Values	$x = -4$	$x = -2.5$	$x = -1.5$	$x = 0$
Sign of $f'(x)$	$f'(-4) = -6 < 0$	$f'(-2.5) \approx 86.6 > 0$	$f'(-1.5) \approx -0.37 < 0$	$f'(0) = 6 > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

Now, by the first derivative test, it is concludable that the function $f(x)$ has a relative minimum at $x = -3, -1$, as the derivative changed its sign from negative to positive and relative maximum at $x = -1$, as the derivative changed its sign from negative to positive.

∎

THE END OF SECTION 4.2

Increasing and Decreasing Functions

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§ 4.3 Concavity and the Second Derivative Test

When examining the graph of a function that forms a curve, we might need to ask whether the curve bends upwards or downwards. It might seem like the question is simply about whether the curve is increasing or decreasing, but it goes beyond that. For example, consider the figures provided below, where both functions are increasing over the interval (a, b), but their shape of increase is different.

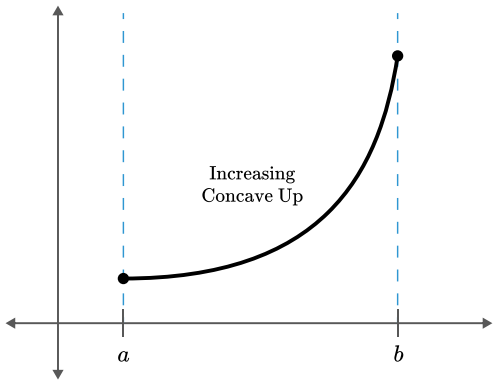

Figure 4.15

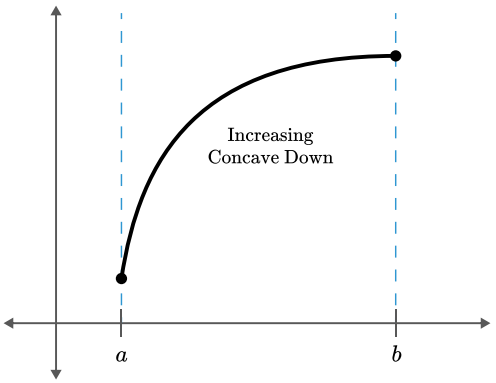

Figure 4.16

The graph of the function on the left is increasing with an *increasing rate*, whereas the graph on the right is increasing with a *decreasing rate*. This is the core idea behind what is referred to as the *concavity* of a function.

The concavity of a function is the direction in which the function curves. It can curve upwards or downwards. For example, the graph in the left figure shown above curves upwards, suggesting that the tangent lines to the curve get steeper and steeper as x increases. This implies that the first derivative of the function is increasing over that interval. When a function's first derivative increases on an interval, the function is said to be *concave up* within that interval.

Similarly, the graph in the right figure shown above curves downwards, suggesting that the tangent lines to the curve get steeper and steeper as x increases. This implies that the first derivative of the function is decreasing over that interval. When a function's first derivative decreases on an interval, the function is said to be *concave down* within that interval.

Summarizing the facts above, we have the following definition of concavity.

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Definition 4.3.1. (Concavity) Let $f(x)$ be a differentiable function over an interval I. Then

1. $f(x)$ is concave up over I, if $f'(x)$ is increasing over I.
2. $f(x)$ is concave down over I, if $f'(x)$ is decreasing over I.

Intuitively, if a function is concave up at a specific point, the graph of the function near that point is above the tangent line drawn at that point. In contrast, if a function is concave down at a specific point, the graph of the function near that point is below the tangent line drawn at that point.

It is important to clearly distinguish between the concepts of increasing/decreasing functions and the concept of concavity, and to do that, refer to the following figure.

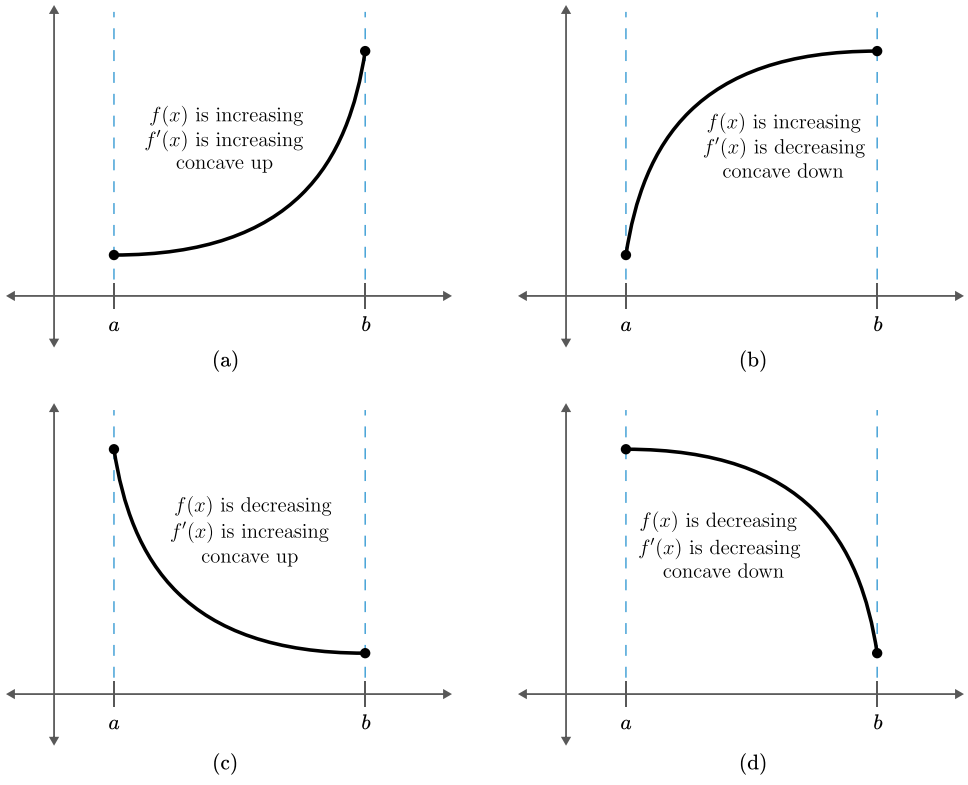

Figure 4.17

In Figure 4.17(a), the function $f(x)$ is increasing and concave up because its first derivative, $f'(x)$, is also increasing. In contrast, in Figure 4.17(b), although the function $f(x)$ is still increasing, but it is concave down, as the derivative $f'(x)$ is decreasing.

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Similarly, in Figure 4.17(c), the function $f(x)$ is decreasing and concave up since $f'(x)$ is increasing. In contrast, in Figure 4.17(d), the function $f(x)$ is decreasing and concave down because $f'(x)$ is decreasing.

Now, to determine if a function is concave up or down over an interval, simply analyze whether its first derivative $f'(x)$ is increasing or decreasing over that interval. It is known from the previous section that if $f'(x) > 0$ over an interval, then $f(x)$ is increasing over that interval. Similarly, if $f'(x) < 0$ over an interval, then $f(x)$ is decreasing over that interval. To determine whether $f'(x)$ itself is increasing or decreasing, we need to compute $f''(x)$ and then apply the same theorem accordingly. That is, if $f''(x) > 0$, then $f'(x)$ is increasing over an interval, and thus $f(x)$ is concave up. In contrast, if $f''(x) < 0$, then $f'(x)$ is decreasing over an interval, and thus $f(x)$ is concave down.

Based on the discussion above, we have the following test for the concavity of a function as follows.

Theorem 4.3.1. (Concavity Test) Let $f(x)$ be a function such that its second derivative exists over an interval I. Then,

1. If $f''(x) > 0$, for all $x \in I$, then $f(x)$ is concave up over I.
2. If $f''(x) < 0$, for all $x \in I$, then $f(x)$ is concave down over I.

Having an understanding of the concavity of a function and a test to determine such, we now consider a point at which the concavity of the function changes from upwards to downwards or vice versa. Such points are referred to as the "Point of Inflection," which is discussed below.

Chap. 4 / Sec. 4.3 / Subsec. 4.3.1 : The Point of Inflection

To understand the Inflection Points consider the graph of the function $f(x) = x^3$, as shown below.

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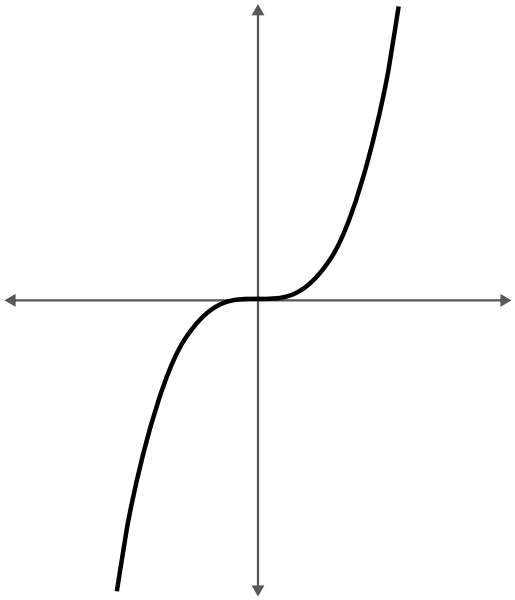

Figure 4.18

It is evident that the function $f(x) = x^3$ increases in an upward shape over the interval $(-\infty, 0)$ and a downward shape over the interval $(0, \infty)$. Thus, the function changes from being concave down to concave up as x passes through the point $x = 0$. It is the point where a change in the concavity of the graph of the function occurred. Such a point is called an *inflection point*.

Inflection points are such points where a curve changes its concavity. That is, at those points, the second derivative changes $f''(x)$ of the function $f(x)$ changes its sign. As a result, we have the following definition.

Definition 4.3.2. (Inflection Points) If a function $f(x)$ is continuous at point $x = a$ and shows a change in concavity at that point, then the point $(a, f(a))$ is a point of inflection of $f(x)$.

Recall that to find the relative extreme values of a function, the first step was to identify its Critical Points. Critical Points are points where the derivative $f'(x)$ equals zero or is undefined. These points act as potential candidates for relative extreme values.

Similarly, to identify potential points of inflection, determine the values of x where the second derivative $f''(x)$ equals zero or does not exist. The following theorem summarizes this process.

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Theorem 4.3.2. (Inflection Points Test)
If c is a point of inflection on the graph of $f(x)$, then either $f''(c) = 0$ or $f''(c) =$ does not exist.

The following example demonstrates the theorem above.

► **Example 4.3.1.** Identify the point of inflection of the function $f(x) = \sin^{-1}(x)$.

Solution. To identify the point of inflection of the function $f(x) = \sin^{-1}(x)$, the first step is to compute the second derivative $f''(x)$ of the function $f(x)$. Therefore,

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\sin^{-1}(x) \\
&= \frac{1}{\sqrt{1-x^2}} \\
f''(x) &= \frac{d}{dx}\frac{1}{\sqrt{1-x^2}} \\
&= \frac{d}{dx}\frac{1}{(1-x^2)^{\frac{1}{2}}} \\
&= \frac{d}{dx}\left(1-x^2\right)^{-\frac{1}{2}} \\
&= -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}} \cdot \frac{d}{dx}\left(1-x^2\right) \\
&= -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}} \cdot (-2x) \\
&= -\frac{1}{\cancel{2}}\frac{-\cancel{2}x}{(1-x^2)^{\frac{3}{2}}} \\
&= \frac{x}{(1-x^2)^{\frac{3}{2}}}
\end{aligned}$$

Now, to find the inflection points, let $f''(x) = 0$. That is,

$$\begin{aligned}
f''(x) &= 0 \\
\frac{x}{(1-x^2)^{\frac{3}{2}}} &= 0 \\
x &= 0 \cdot \left(1-x^2\right)^{\frac{3}{2}} \\
x &= 0
\end{aligned}$$

This suggests that $f''(x)$ changes its sign from negative to positive at $x = 0$. This is because denominator of $f''(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$ is positive on the entire interval $(-1, 1)$, however, the numerator is negative on $(-1, 0)$ and positive on $(0, 1)$,

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making $f''(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$ negative on $(-1, 0)$ and positive on $(0, 1)$. Therefore, $f(x)$ is concave down on $(-1, 0)$ and concave up on $(0, 1)$, making $x = 0$ is a inflection point of $f(x)$. ∎

Chap. 4 / Sec. 4.3 / Subsec. 4.3.2 : The Second Derivative Test

Previously, it was discussed how the second derivative can be used to determine whether a function is curving upwards or downwards. Additionally, the second derivative can also be used as a test to identify relative maximum and minimum value of a function.

Theorem 4.3.3. (The Second Derivative Test)
Suppose a function $f(x)$ such that $f''(x)$ exists over an interval containing c. Then,

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a relative maximum at c.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a relative minimum at c.
3. If $f'(c) = 0$ and $f''(c) = 0$, then $f(x)$ may have a relative maximum, minimum, or neither at c.

———————The End of Section 4.3
Concavity and the Second Derivative Test

§ 4.4 Rolle's Theorem and The Mean Value Theorem

The Mean Value is serves as a crucial theorem of calculus. Numerous theorem and application of calculus rely on this theorem. In this section, we discuss the Mean Value Theorem by first discussing a special case of it: Rolle's Theorem.

Chap. 4 / Sec. 4.4 / Subsec. 4.4.1 : Rolle's Theorem

Rolle's Theorem is a fundamental result in differential calculus and is really common sense when seen visually. For example, consider a function $f(x)$ such that it is continuous over a closed interval $[a, b]$ and differentiable over an open

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interval (a, b) with $f(a) = f(b)$. Then there exists a point c between a and b such that the derivative at c is 0. That is, the slope of the tangent line at that point is 0, as shown below.

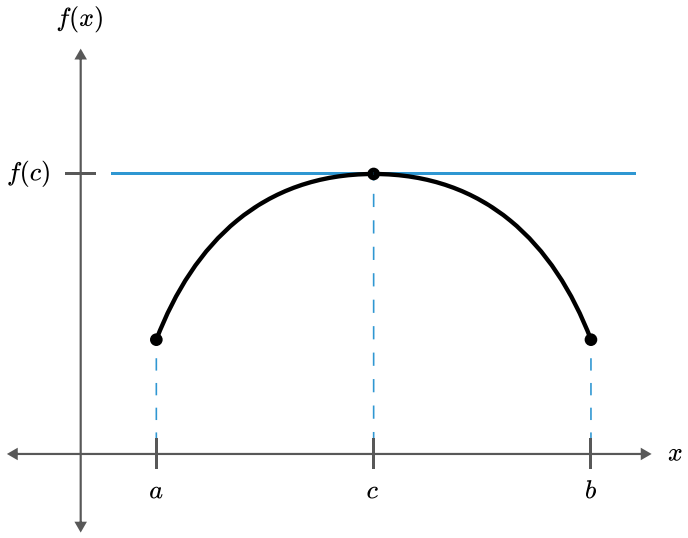

Figure 4.19

The basic idea is that if a smooth curve starts and ends at the same height (that is, if $f(a) = f(b)$), it must have at least one point where the slope of the curve flattens out (i.e., the derivative becomes zero). This occurs because, intuitively, the function has to "turn around" somewhere between a and b to return to the same value at both endpoints. Therefore, we have the following theorem.

Theorem 4.4.1. (Rolle's Theorem) Let $f(x)$ be a function meeting the following conditions,

1. $f(x)$ is continuous over $[a, b]$
2. $f(x)$ is differentiable over (a, b)
3. $f(a) = f(b)$

then there must exist at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let $f(a) = f(b) = k$ and consider the following cases,

1. Case 1: $f(x) = k$, for all $x \in (a, b)$
2. Case 2: $f(x) > k$, for some $x \in (a, b)$
3. Case 3: $f(x) < k$, for some $x \in (a, b)$

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Case 1: Suppose, $f(x) = k$, for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$, as $f(x)$ is a constant function.

Case 2: Suppose, $f(x) > k$, for some $x \in (a, b)$. Since $f(x)$ is continuous over $[a, b]$, therefore, by the Extreme Value Theorem, it has an absolute maximum over $[a, b]$. Given that $f(a) = f(b)$, the absolute maximum cannot occur at the endpoints a or b. Thus, the absolute maximum must occur at some interior point c in (a, b).
Since all absolute maximums are also relative maximums, therefore $f(x)$ must have a relative maximum at some point $c \in (a, b)$. Now, according to the Relative/Local Extreme Value Theorem, if $f(x)$ has a relative maximum at c and is differentiable at c, then $f'(c) = 0$.

Case 3: Suppose, $f(x) < k$, for some $x \in (a, b)$. Since $f(x)$ is continuous over $[a, b]$, therefore, by the Extreme Value Theorem, it has an absolute minimum over $[a, b]$. Given that $f(a) = f(b)$, the absolute minimum cannot occur at the endpoints a or b. Thus, the absolute maximum must occur at some interior point c in (a, b).
Since all absolute minimums are also relative minimums, therefore $f(x)$ must have a relative minimum at some point $c \in (a, b)$. Now, according to the Relative/Local Extreme Value Theorem, if $f(x)$ has a relative minimum at c and is differentiable at c, then $f'(c) = 0$. ∎

The theorem above can be applied in certain situations very cleverly. For instance, consider the following example.

► **Example 4.4.1.** Show that $f(x) = x^3 + x + 1$ has exactly one real solution.

Solution. Notice that the value of the function $f(x) = x^3 + x + 1$ is -5 and 1 for $x = -2$ and $x = 0$, respectively. Therefore, it follows from the Intermediate Value Theorem (IVT) that there exists a point k between $x = -2$ and $x = 0$ such that $f(k) = 0$.

Now, besides of that single point k, if there were two points $x = a$ and $x = b$ for which $f(a) = f(b) = 0$, then the Rolle's theorem would guarantee that there exists a point $x = c$ between the points $x = a$ and $x = b$ for which $f'(c) = 0$. However, this can never happen as the derivative of $f(x) = x^3 + x + 1$ is,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(x^3 + x + 1 \right)$$

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$$= 3x^2 + 1 > 0$$

Which is always greater than 0 for all x. Therefore, Rolle's theorem cannot be met and there doesn't exist any point other than one single $x = k$, for which $f(k) = 0$. ∎

Chap. 4 / Sec. 4.4 / Subsec. 4.4.2 : The Mean Value Theorem (MVT)

The Mean Value Theorem can best be understood visually. For example, consider the figure below.

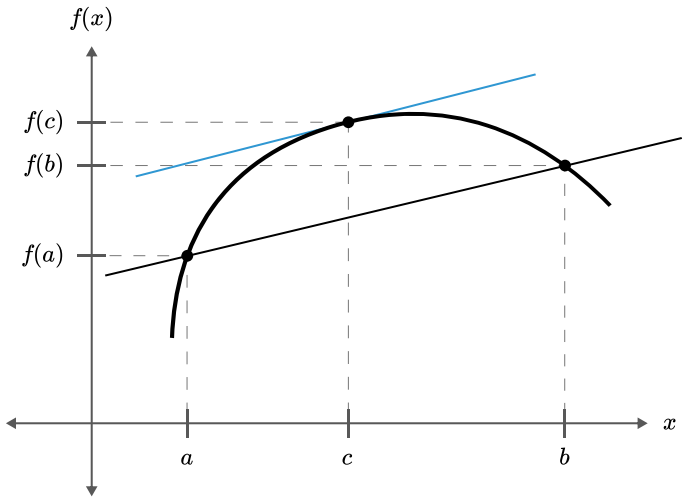

Figure 4.20

In the figure above, the slope of the secant line that passes through the point $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}$$

Now, the Mean Value Theorem essentially states that there exists at least one point $c \in (a, b)$ such that the slope of the tangent line at c is equal to the slope of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$. That is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This implies that there is a point c where the tangent line at that point c is parallel to the secant line that passes through the $(a, f(a))$ and $(b, f(b))$. Therefore, we have the following theorem.

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Theorem 4.4.2. (The Mean Value Theorem)
Let $f(x)$ be a function meeting the following conditions,

1. $f(x)$ is continuous over $[a, b]$
2. $f(x)$ is differentiable over (a, b)

then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(b) = f'(c)(b - a)$$

Proof. Consider a straight line that goes through the points $(a, f(a))$ and $(b, f(b))$. The slope of this line is,

$$m = \frac{f(b) - f(a)}{b - a}$$

Since, the line goes through the point $(a, f(a))$ and has a slope $m = \dfrac{f(b) - f(a)}{b - a}$, therefore the point-slope equation of the line yields,

$$y - f(a) = m(x - a)$$

or,

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Now, let a new function $g(x)$ be the difference between the function $f(x)$ and the line y as follows,

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right) \\ &= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \end{aligned}$$

Now, if $x = a$, then $g(x)$ yields,

$$\begin{aligned} g(x) &= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \\ &= f(a) - \frac{f(b) - f(a)}{b - a}\cancelto{0}{(a - a)} - f(a) \\ &= f(a) - f(a) \end{aligned}$$

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$$= 0$$

If $x = b$, then $g(x)$ yields,

$$\begin{aligned} g(x) &= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \\ &= f(b) - \frac{f(b) - f(a)}{\cancel{b - a}}\cancel{(b - a)} - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) \\ &= f(b) - f(b) + f(a) - f(a) \\ &= 0 \end{aligned}$$

This implies that $g(a) = 0 = g(b)$. Now, since $f(x)$ and the straight line y are a differentiable function over (a, b), therefore $g(x)$ is also a differentiable function over (a, b). Furthermore, since $f(x)$ and y are a continuous function over $[a, b]$, therefore $g(x)$ is also a continuous function over $[a, b]$. Thus, $g(x)$ satisfies Rolle's Theorem implying that there exists a point $c \in (a, b)$ for which $g'(c) = 0$. Therefore, let us differentiate the function $g(x)$,

$$\begin{aligned} g'(x) &= \frac{d}{dx} g\,(x) \\ &= \frac{d}{dx}\left(f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)\right) \\ &= \frac{d}{dx} f(x) - \frac{d}{dx}\left(\frac{f(b) - f(a)}{b - a}(x - a)\right) - \frac{d}{dx} f(a) \\ &= f'(x) - \frac{f(b) - f(a)}{b - a} \cdot \frac{d}{dx}\,(x - a) \\ &= f'(x) - \frac{f(b) - f(a)}{b - a} \cdot 1 \\ \therefore g'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Since, $g(x)$ satisfies Rolle's Theorem suggesting that there exists a point $c \in (a, b)$ for which $g'(c) = 0$, therefore we let,

$$\begin{aligned} g'(c) &= 0 \\ f'(c) - \frac{f(b) - f(a)}{b - a} &= 0 \\ \therefore f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

This completes the proof. ∎

A few examples to demonstrate the theorem above are provided below.

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► **Example 4.4.2.** Test the Mean Value Theorem for $f(x) = x^2$ on the interval $[1, 4]$.

Solution. The function $f(x) = x^2$ is a polynomial, therefore it is continuous and differentiable everywhere.

Now, the slope of the secant line that passes through the points $x = 1$ and $x = 4$ is,

$$\begin{aligned}\frac{f(4) - f(1)}{4 - 1} &= \frac{16 - 1}{4 - 1} \\ &= \frac{15}{3} \\ &= 5\end{aligned}$$

Now, to test the Mean Value Theorem, we need to find a point c such that $f'(c) = 5$. Therefore, we differentiate $f(x) = x^2$ as follows.

$$\frac{d}{dx}x^2 = 2x$$

Setting $f'(c) = 5$, we get,

$$\begin{aligned}2c &= 5 \\ \therefore c &= \frac{5}{2} \\ &= 2.5\end{aligned}$$

Thus, at $x = 2.5$, the function $f(x) = x^2$ has a tangent line where its slope matches the secant line that passes through the point $x = 1$ and $x = 4$. ∎

► **Example 4.4.3.** Test the Mean Value Theorem for $f(x) = \sin(x)$ on the interval $[0, \pi]$.

Solution. The function $f(x) = \sin(x)$ is both continuous and differentiable on the interval $[0, \pi]$.

The slope of the secant line passing through the points $x = 0$ and $x = \pi$ is,

$$\frac{\sin(\pi) - \sin(0)}{\pi - 0} = \frac{0 - 0}{\pi}$$

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$$= 0$$

Now, to test the Mean Value Theorem, we need to find a point c such that $f'(c) = 0$. Therefore,

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Settings $f'(c) = 0$, we get,

$$\begin{aligned}\cos(c) &= 0\\ \therefore c &= \cos^{-1}(0)\\ &= \frac{\pi}{2}\end{aligned}$$

Thus, at $x = \frac{\pi}{2}$, the function $f(x) = \sin(x)$ has a tangent line where its slope matches the secant line that passes through the points $x = 0$ and $x = \pi$. ∎

———————The End of Section 4.4

Rolle's Theorem and The Mean Value Theorem

§ 4.5 Related Rates

In Chapter 3, Section 3.3, the Chain Rule was introduced. In this section, we learn to apply the Chain Rule in a way that shows how the rate of change of multiple variables relates to each other as they change over time. This is rooted in the understanding that if two or more related variables are changing over time, then the rate at which they are changing is also related. For example, suppose the variables x and y are both changing over time. The relation between x and y is given by the equation $y = 2x^3$. Now, given that the rate of change of x with respect to time t is n, when $x = a$, is it possible to find the rate of change of y with respect to t, when $x = a$? The answer is positive, because we know that

$$\begin{aligned}y &= 2x^3\\ \frac{d}{dt}(y) &= \frac{d}{dt}\left(2x^3\right)\\ \frac{dy}{dt} &= 6x^2 \cdot \frac{dx}{dt}\end{aligned}$$

Now, since the rate of change of x with respect to t, when $x = a$, is n; therefore $\frac{dx}{dt} = n$. Thus, the rate of change of y with respect to t, when $x = a$, is,

$$\frac{dy}{dt} = 6a^2n$$

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So, the rate of change of y with respect to t, when $x = a$, is $6a^2n$. It is interesting to note how we have figured out the rate of change of one variable in terms of the other related variable, which was known to us.

In the real world, there are numerous examples of related variables that changes over time. For example, consider inflating a balloon. As the balloon gets inflated with respect to time, it becomes evident that its size expands, indicating an increase in its volume V. Now, assuming the shape of the inflated balloon is spherical and knowing the formula for the volume of a sphere, that is, $V = \frac{4}{3}\pi r^3$, it can be said that as the volume increases over time, so does the radius r. This implies that the rate of change in the volume, that is $\frac{dV}{dt}$ is related to the rate of change in the radius $\frac{dr}{dt}$ since V is related to r by the equation $V = \frac{4}{3}\pi r^3$. We shall use these relations to solve the following problem.

► **Example 4.5.1.** A circle's radius is increasing at a constant rate of 5 cm per second. How fast is the area of the circle increasing when the radius is 100 cm?

Solution. Let r be the radius of the circle and A be the area of the circle at a time t, then we're given $\frac{dr}{dt} = 2\ \frac{\text{cm}}{\text{s}}$. We need to find $\frac{dA}{dt}$, when $r = 100$ m.

In order to do that notice first that the area of the circle A and the radius r of the circle is related by the following equation.

$$A = \pi r^2$$

Now, differentiating both sides of the equation above yields,

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}\left(\pi r^2\right)\\ &= \pi\frac{d}{dt}r^2\\ &= \pi \cdot 2r\frac{dr}{dt}\\ &= 2r\pi\frac{dr}{dt}\end{aligned}$$

Now, substituting the given $\frac{dr}{dt} = 2\ \frac{\text{cm}}{\text{s}}$ and $r = 100\text{m}$ into the equation above yields,

$$2r\pi\frac{dr}{dt} = 2\pi\,(100)\,(2)$$

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$$= 2\pi\,(100)\,(2)$$
$$\approx 1256.64\ \frac{\text{cm}^2}{\text{s}}$$

Therefore, the area of the circle is increasing at a rate of approximately $1256.64\ \frac{\text{cm}^2}{\text{s}}$, when the radius is 100 cm. ∎

► **Example 4.5.2.** A cylindrical tank with a radius of 2 meters is draining water at a rate of 3 cubic meters per minute. How fast is the water level dropping when the height of the water is 5 meters?

Solution. Let h be the height of the water, and V the volume of water in the tank. We are given $\frac{dV}{dt} = -3\ \frac{\text{m}^3}{\text{s}}$ ("-3" is due to the reason that the volume is decreasing) and asked to find $\frac{dh}{dt}$ when $h = 5\text{m}$.

In order to do that notice that the volume V and the height h are related by the following equation.

$$V = \pi r^2 h$$

Since the radius $r = 2\text{m}$ is constant, therefore, the equation above is simplified as follows.

$$V = \pi r^2 h$$
$$= \pi 2^2 h$$
$$= 4\pi h$$

Now, differentiating both sides of the equation with respect to time t above yields,

$$\frac{dV}{dt} = \frac{d}{dt}\,(4\pi h)$$
$$= 4\pi \frac{dh}{dt}$$

Since we are given $\frac{dV}{dt} = -3\ \frac{\text{m}^3}{\text{s}}$, therefore,

$$\frac{dV}{dt} = 4\pi \frac{dh}{dt}$$
$$-3 = 4\pi \frac{dh}{dt}$$

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$$\begin{aligned}\frac{dh}{dt} &= \frac{-3}{4\pi} \\ &\approx -0.24\ \frac{\text{m}}{\text{s}}\end{aligned}$$

Therefore, the water level is dropping at the rate of approximately $0.24\ \frac{\text{m}}{\text{s}}$. ∎

► **Example 4.5.3.** Sand falls from a conveyor belt at a rate of 1.5 cubic meters per minute. This forms a conical pile of sand. The radius of the base of this conical pile is always half the height. How fast is the height of the pile increasing when the height is 3 meters?

Solution. Let V be the volume of the cone and h be the height of the cone. We are given $\frac{dV}{dt} = 1.5\ \frac{\text{m}^3}{\text{min}}$ and $r = \frac{h}{2}$, and asked to find $\frac{dh}{dt}$ when $h = 3\text{m}$.

In order to do that notice that the volume V and the height h are related by the volume of a right circular cone formula. That is,

$$V = \frac{1}{3}\pi r^2 h$$

Since the radius of the pile is always half the height, that is, $r = \frac{h}{2}$, therefore,

$$\begin{aligned}V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{12}\pi h^3\end{aligned}$$

Now, differentiating both sides of the equation above with respect to time t yields,

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) \\ &= \frac{1}{12}\pi\frac{d}{dt}h^3 \\ &= \frac{1}{\cancel{12}}\pi \cdot \cancel{3}h^2\frac{dh}{dt} \\ &= \frac{1}{4}\pi h^2\frac{dh}{dt}\end{aligned}$$

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Since $\dfrac{dV}{dt} = 1.5 \dfrac{\text{m}^3}{\text{min}}$ and $h = 3\text{m}$,

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi}{4}h^2\frac{dh}{dt}\\ 1.5 &= \frac{\pi}{4}(3)^2\frac{dh}{dt}\\ \frac{dh}{dt} &= \frac{6}{9\pi}\\ &= \frac{2}{3\pi}\ \frac{\text{m}}{\text{min}}\end{aligned}$$

Therefore, The height of the sand pile is increasing at a rate of $\dfrac{2}{3\pi}\ \dfrac{\text{m}}{\text{min}}$ or approximately $0.212\ \dfrac{\text{m}}{\text{min}}$. ∎

► **Example 4.5.4.** A 10-foot ladder is leaning against a wall. The bottom of the ladder is being moved away from the wall at a speed of 2 feet per second. At the moment when the bottom of the ladder is 5 feet away from the wall, how fast is the top of the ladder sliding down the wall?

Solution. Let x be the distance from the bottom of the ladder to the wall and y be the height of the top of the ladder on the wall. We are given $\dfrac{dx}{dt} = 2\ \dfrac{\text{ft}}{\text{s}}$ and asked to find $\dfrac{dy}{dt}$ when $x = 5\text{ft}$.

In order to do that notice that the ladder forms a right triangle and the variables are related by the Pythagorean theorem. That is,

$$x^2 + y^2 = 10^2$$

Now, differentiating both sides of the equation above with respect to time t yields,

$$\begin{aligned}\frac{d}{dt}x^2 + \frac{d}{dt}y^2 &= \frac{d}{dt}100\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0\\ x\frac{dx}{dt} + y\frac{dy}{dt} &= 0\end{aligned}$$

Since $x = 5\text{ft}$, therefore, the value of y can be found using the Pythagorean theorem as follows.

$$x^2 + y^2 = 100$$

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$$\begin{aligned} y^2 &= 100 - x^2 \\ y &= \sqrt{100 - 5^2} \\ &= 75\text{ft} \end{aligned}$$

Now, substituting $x = 5\text{ft}$, $y = 75\text{ft}$, and $\dfrac{dx}{dt} = 2\ \dfrac{\text{ft}}{\text{s}}$ yields,

$$\begin{aligned} x\frac{dx}{dt} + y\frac{dy}{dt} &= 0 \\ 5 \cdot 2 + 75\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{10}{75}\ \frac{\text{ft}}{\text{s}} \end{aligned}$$

Therefore, the top of the ladder is sliding down at a rate of $\dfrac{10}{75}\ \dfrac{\text{ft}}{\text{s}}$, or approximately $0.13\ \dfrac{\text{ft}}{\text{s}}$. ∎

Having discussed the related rates problems through examples, the following can be thought of as a general guideline for solving such problems.

1. Identify all given variables and the variables that are to be determined. Draw a figure if applicable
2. An equation that includes the variables for which the rates of change are either given or need to be found.
3. Use the Chain Rule and differentiate implicitly both sides of the equation with respect to time.
4. Substitute all known values into the equation, then solve for the unknown rate of change. respect to time

———————The End of Section 4.5
Related Rates

§ 4.6 Linearization and Differentials

In this section, we discuss a method by which a complicated function can be approximated by a relatively simper function that is easier to work with, up to our desired level of accuracy. This method is commonly referred to as the

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"linearization/linear approximation/tangent line approximation," and is based on the observation that when we "zoom" into the graph of a smooth curve at a point P that has a tangent line drawn at that point, the curve looks more and more like the tangent line drawn at that point. Therefore, let us start our discussion of the linearization as follows.

Chap. 4 / Sec. 4.6 / Subsec. 4.6.1 : Linearization

We are all familiar with how digital calculators have significantly enhanced our lives, making tasks easier and more efficient than ever. Nevertheless, this invaluable tool was nonexistent just a few centuries ago. Is it thoughtful enough for us to think about how individuals in earlier centuries dealt with the calculation of values such as $\sqrt{4.01}$? Think about it! What really is the value of $\sqrt{4.01}$ or at least a really good approximation of it? The answer isn't readily apparent. However, the derivatives can be used to deal with such problems at hand. For that, let us recall that $\sqrt{4} = 2$. Since $4.01 > 4$ therefore the value of $\sqrt{4.01} > \sqrt{4}$. So, we write $\sqrt{4.01} = 2+r$, where $r \in \mathbb{R}$, $r > 0$. Now, the problem at hand is to determine the sufficient value of r such that $2 + r \approx \sqrt{4.01}$. For that, let us draw a tangent line to the graph of $f(x) = \sqrt{x}$, at the point $x = 4$. The drawn tangent line is reasonably close to the actual function near the point $x = 4$.

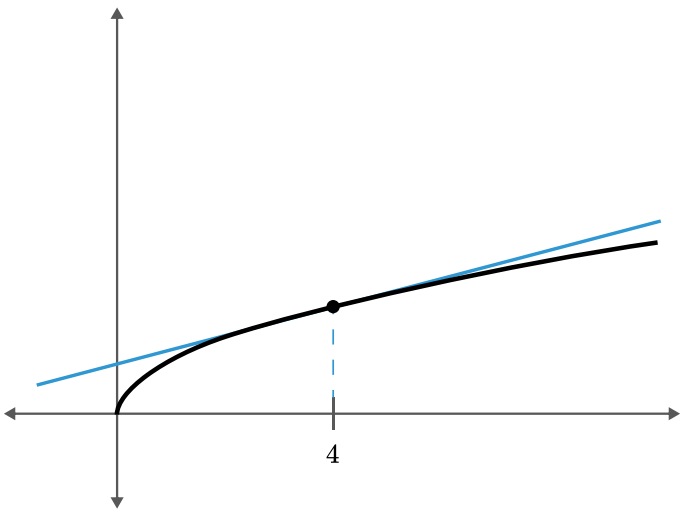

Figure 4.21

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Since the tangent line is a straight line, therefore it has a slope of

$$\begin{aligned} m &= \frac{\sqrt{4.01} - \sqrt{4}}{4.01 - 4} \\ &= \frac{\sqrt{4.01} - \sqrt{4}}{0.01} \end{aligned}$$

Now, multiplying both sides by 0.01 and then solving for $\sqrt{4.01}$ yields,

$$\begin{aligned} 0.01 \times m &= \frac{\sqrt{4.01} - \sqrt{4}}{\cancel{0.01}} \times \cancel{0.01} \\ 0.01 \times m &= \sqrt{4.01} - \sqrt{4} \\ 0.01m + \sqrt{4} &= \sqrt{4.01} \\ \therefore \sqrt{4.01} &= 2 + 0.01m \end{aligned}$$

Now, a value of r is found to be $r = 0.01m$, where m is the slope of the tangent line at the point $x = 4$. It is known that for a straight line, the slope is equal to its derivative, that is, slope $= m = f'(x)$. Therefore, the derivative of $f(x) = \sqrt{x}$ at the point $x = 4$ yields,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sqrt{x} \\ &= \frac{1}{2\sqrt{x}} \\ f'(4) &= \frac{1}{2\sqrt{4}} \\ &= \frac{1}{2 \cdot 2} \\ &= \frac{1}{4} \\ &= m \end{aligned}$$

This implies that

$$\begin{aligned} \sqrt{4.01} &= 2 + r \\ &= 2 + 0.01m \\ &= 2 + \left(0.01 \cdot \frac{1}{4}\right) \\ &= 2 + 0.0025 \\ &= 2.0025 \end{aligned}$$

This is reasonably a good approximation. We now generalize this idea further below.

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Consider a function $y = f(x)$. Draw a tangent line to $y = f(x)$ at the point $x = a$, as shown below.

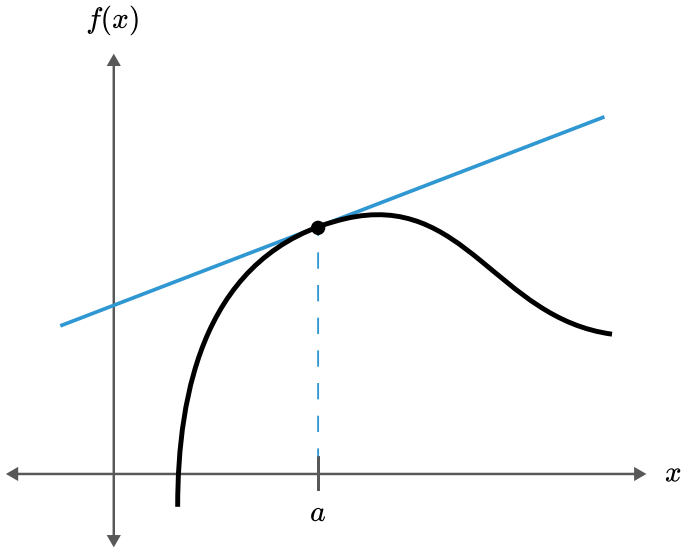

Figure 4.22

The slope of the tangent line is $m = \dfrac{y - f(a)}{x - a}$. Multiplying both sides by $(x-a)$ and solving for y yields,

$$\begin{aligned} m &= \frac{y - f(a)}{x - a} \\ (x - a) \cdot m &= \frac{y - f(a)}{\cancel{x - a}} \cdot \cancel{(x - a)} \\ (x - a) \cdot m &= y - f(a) \\ (x - a) \cdot m + f(a) &= y \\ \therefore y &= f(a) + m\,(x - a) \end{aligned}$$

It is known that for a linear function, the slope remains constant across all points, therefore, the slope is equivalent to the derivative at any given point. Thus,

$$y = f(a) + f'(a)(x - a)$$

This implies that the tangent line to the graph of the function at the point $x = a$ represents the graph of the following linear function.

$$L(x) = f(a) + f'(a)(x - a)$$

Thus, we have the following definition.

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Definition 4.6.1. (Linearization) If $f(x)$ is a differentiable function at $x = a$, then the linearization of $f(x)$ at a is defined as

$$L(x) = f(a) + f'(a)(x - a)$$

Where, $L(x) \approx f(x)$ for values of $f(x)$ near a.

A few examples are provided below.

► **Example 4.6.1.** What is the linearization of $f(x) = \sin(x)$ at $x = 0$?

Solution. The derivative of $f(x) = \sin(x)$ is,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sin(x) \\ &= \cos(x) \end{aligned}$$

Now, since $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$, therefore the linearization at $x = 0$ yields,

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= f(0) + f'(0)(x - 0) \\ &= 0 + 1(x) \\ &= x \end{aligned}$$

This implies that $\sin(x) \approx x$, near $x = 0$. Therefore, the linearization of $f(x) = \sin(x)$, at the point $x = 0$ is $L(x) = x$. In other words, if the graphs of the functions $y = \sin(x)$ and $y = x$ are zoomed in at the point $x = 0$, they become indistinguishable. ▮

► **Example 4.6.2.** What is the linearization of $f(x) = 2\sqrt{1 + x}$ at $x = 3$?

Solution. The derivative of $f(x) = 2\sqrt{1 + x}$ is,

$$\begin{aligned} f'(x) &= \frac{d}{dx} 2\sqrt{1 + x} \\ &= 2 \cdot \frac{d}{dx}(1 + x)^{\frac{1}{2}} \\ &= 2 \cdot \frac{1}{2}(1 + x)^{-\frac{1}{2}} \end{aligned}$$

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$$= \frac{1}{(1+x)^{\frac{1}{2}}}$$
$$= \frac{1}{\sqrt{1+x}}$$

Since, $f(3) = 2\sqrt{1+3} = 2\sqrt{4} = 4$, and $f'(3) = \frac{1}{\sqrt{1+3}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$, therefore the linearization at $x = 3$ yields,

$$\begin{aligned} L(x) &= f(a) + f'(a)\,(x-a) \\ &= f(3) + f'(3)\,(x-3) \\ &= 4 + \frac{1}{2}\,(x-3) \end{aligned}$$

This implies that $2\sqrt{1+x} \approx 4 + \frac{1}{2}\,(x-3)$, near $x = 3$. Therefore, the linearization of $f(x) = 2\sqrt{1+x}$, at the point $x = 3$ is $L(x) = 4 + \frac{1}{2}\,(x-3)$. In other words, if the graphs of the functions $y = 2\sqrt{1+x}$ and $y = 4 + \frac{1}{2}\,(x-3)$ are zoomed in at the point $x = 3$, they become indistinguishable. ∎

Having discussed the linearization, we now shift our focus to discuss the differentials that approximates the amount a function changes.

Chap. 4 / Sec. 4.6 / Subsec. 4.6.2 : Differentials

As of yet, we have seen that linearization can be used to approximate a function's values near a point $x = a$. However, they can also be used to approximate the amount a function changes as a result of a small change in the input x. To see how, recall that,

$$\begin{aligned} f(x) &\approx L(x) \\ &= f(a) + f'(a)(x-a) \\ \therefore f(x) &\approx f(a) + f'(a)(x-a) \end{aligned}$$

where, a is a fixed point and x is its nearby points. This implies that

$$\begin{aligned} f\,(x) &\approx f\,(a) + f'\,(a)\,(x-a) \\ f\,(x) - f\,(a) &\approx f'\,(a)\,(x-a) \\ \Delta y &\approx f'\,(a)\,\Delta x \end{aligned} \tag{1}$$

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where, $\Delta x = x - a$ represents a change in the x-coordinate between a and x and $\Delta y = f(x) - f(a)$ represents a corresponding change in the y-coordinate between $f(a)$ and $f(x)$, respectively. This suggests that a change in y can be approximated by the change in x multiplied by the derivative of the function at that point.

Now, suppose a function $y = f(x)$ has changed from $x = a$ to $x = a + \Delta x$. Then the exact change in y, that is, Δy, yields the following.

$$\Delta y = f(a + \Delta x) - f(a)$$

Now, sometimes it is more difficult to calculate the exact change in y, that is, Δy, compared to just approximating it by the use of linearization as discussed in the previous subsection. That is, for all x near a, $f(x)$ can more easily be approximated by,

$$L(x) = f(a) + f'(a)(x - a)$$

Now, the exact change in L as x changes from $x = a$ to $x = a + \Delta x$, is

$$\begin{aligned}
\Delta L &= L(a + \Delta x) - L(x) \\
&= \underbrace{\Big(f(a) + f'(a)(a + \Delta x - a)\Big)}_{L(a+\Delta x)} - \underbrace{\Big(f(a) + f'(a)(a - a)\Big)}_{L(x)} \\
&= \Big(f(a) + f'(a)\Delta x\Big) - f(a) \\
&= f'(a)\Delta x \\
\therefore \Delta L &= f'(a)\Delta x \qquad (1)
\end{aligned}$$

Note that ΔL is an approximation of the actual change Δy, as x changes from $x = a$ to $x = a + \Delta x$. This approximation gets better and better as $\Delta x \to 0$, that is, as the point $x = a + \Delta x$ gets closer and closer to $x = a$.

Now, based on this approximation ΔL of the actual change Δy, the concept of differentials can be defined precisely. Historically, the differential of a function (denoted as dy) was thought of as an infinitely small (or infinitesimal) change in y, corresponding to an infinitely small (or infinitesimal) change in x (denoted as dx), if we were to put it in an oversimplified manner. When these infinitesimally small changes are compared together by a ratio of $\frac{dy}{dx}$, the instantaneous rate of change (i.e., the derivative) of a function is found. Nevertheless, defining such infinitesimal quantities dy and dx seemed to be all based on intuition, which

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lacked the mathematical rigor.

Now, centuries later, these differentials can indeed be defined rigorously with the concept of linearization. For that, consider the following figure, where the change in the function $y = f(x)$ as x changed from $x = a$ to $x = a + dx$ is approximated by the tangent line drawn at $x = a$, using linearization.

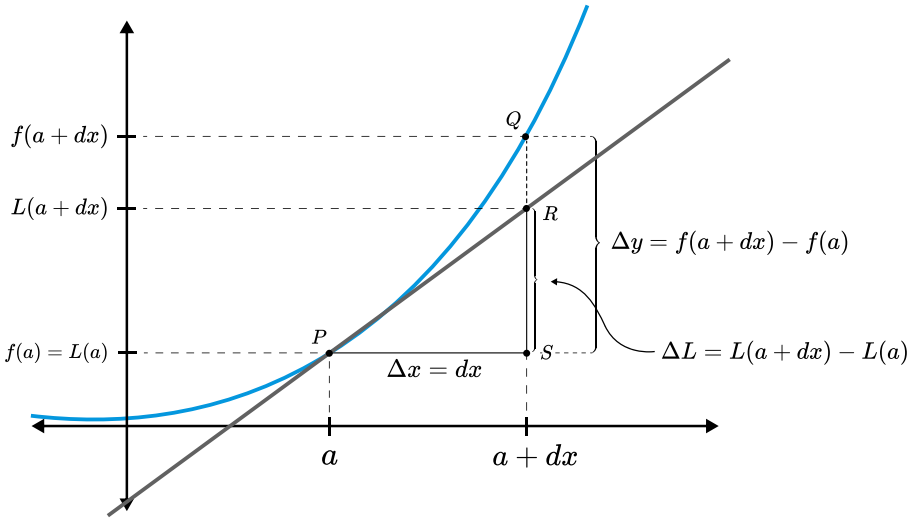

Figure 4.23

In the figure above, as x changes from $x = a$ to $x = a + dx$, where $dx \in \mathbb{R}$ is a small increment, there occur two changes. One change occurred in the function $y = f(x)$ (denoted as Δy), and the other one is in the tangent line $L(x)$ (denoted as ΔL). Here, Δy is the actual change and ΔL serves as its approximation.

Now, the differential dy can be defined as the change in this linearization, that is, ΔL. This implies that

$$\begin{aligned} dy &= \Delta L \\ &= f'(a)\,\Delta x && \text{from equation (1)} \\ &= f'(a)\,\underbrace{\big((x+dx) - x\big)}_{\Delta x} \\ &= f'(a)\,dx && \text{since } \Delta x = dx = \text{real number} \end{aligned}$$

This implies that at $x = a$, the differential dy is $dy = f'(a)dx$. Therefore, more generally, at any variable point x, the differential dy yields,

$$dy = f'(x)\,dx$$

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Thus, we finally have the following definition of the differential of y.

Definition 4.6.2. (Differentials) If $y = f(x)$ is a differentiable function, and the differential of x (denoted as dx) is not 0, then the differential of y (denoted as dy) is,

$$dy = f'(x)\,dx$$

This definition aligns with the old intuition that when $dy \div dx$, the derivative is found. That is,

$$\begin{aligned} dy &= f'(x)\,dx \\ \frac{dy}{dx} &= \frac{f'(x)\cancel{dx}}{\cancel{dx}} \\ \frac{dy}{dx} &= f'(x) \end{aligned}$$

This is perhaps the point where confusions come into play! Previously, $\dfrac{dy}{dx}$ was understood as the derivative of a function, defined as the limit of the quotient $\dfrac{\Delta y}{\Delta x}$, as $\Delta x \to 0$. In other words,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\dfrac{dy}{dx}$ was merely a notation and not an actual quotient of dy over dx. However, now that we have defined meanings for dy and dx, the derivative of a function $f'(x)$ can indeed be expressed as a quotient of dy and dx. That is,

$$\begin{aligned} dy \div dx &= \frac{dy}{dx} \\ &= \frac{f'(x)\cancel{dx}}{\cancel{dx}} \\ &= f'(x) \\ &= \text{the derivative of } f(x) \end{aligned}$$

This suggests that,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx} \\ dy &= \frac{dy}{dx} dx \end{aligned}$$

Notice how dx's in the equation above cannot simply be canceled out. This is because $\dfrac{dy}{dx}$ on the left side of the equation represents the derivative of a function,

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and thus it is merely a notation. In other words,

$$dy = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot dx$$

Note that every differential rule summarized before can be written in differential form. That is,

1. $d(c) = 0$
2. $d(x) = dx$
3. $d(x^n) = nx^{n-1}dx$
4. $d\left(\frac{x^{n+1}}{n+1}\right) = x^n dx$
5. $d(\sqrt{x}) = \frac{1}{2\sqrt{x}}dx$
6. $d\left(e^x\right) = e^x dx$
7. $d\left(e^{-x}\right) = -e^{-x}dx$
8. $d\left(a^x\right) = a^x \ln(a)dx$
9. $d\left(\ln(x)\right) = d\left(\log_e(x)\right) = \frac{1}{x}dx$
10. $d\left(\log_a(x)\right) = \frac{1}{x\ln(a)}dx$
11. $d\left(\sin(x)\right) = \cos(x)dx$
12. $d\left(\cos(x)\right) = -\sin(x)dx$
13. $d\left(\tan(x)\right) = \sec^2(x)dx$
14. $d\left(\cot(x)\right) = -\csc^2(x)dx$
15. $d\left(\sec(x)\right) = \sec(x)\tan(x)dx$
16. $d\left(\csc(x)\right) = -\csc(x)\cot(x)dx$
17. $d\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}dx$
18. $d\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}dx$
19. $d\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}dx$
20. $d\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}dx$
21. $d\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}dx$
22. $d\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}dx$

The End of Section 4.6

Linearization and Differentials

§ 4.7 Newton's Method

Newton's Method is a numerical method to approximate the roots of a function. A root of a function is a value of x that satisfies the equation $f(x) = 0$. Graphically, the root represents the x-intercept of the graph of the function $y = f(x)$. It is known from algebra that the root of a first-degree polynomial, $ax + b = 0$, is determined by the formula:

$$x = -\frac{b}{a}$$

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Similarly, the roots of a second-degree polynomial, $ax^2 + bx + c = 0$, are determined by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Formulas also exist to find the roots of third-degree and fourth-degree polynomials, but they aren't as straightforward as the ones above. In fact, it is not possible to derive a general formula for fifth-degree polynomials or polynomials of any higher degree. As a result, if we ever need to find the roots of an equation such as the following,

$$x^5 + x^4 + x^3 + x^2 + x$$

It may best be approximated numerically using the method discussed in this section.

Newton's Method is one of many methods by which the roots of a function can be approximated. The method can best be described in a graphically. For that, consider the figure below.

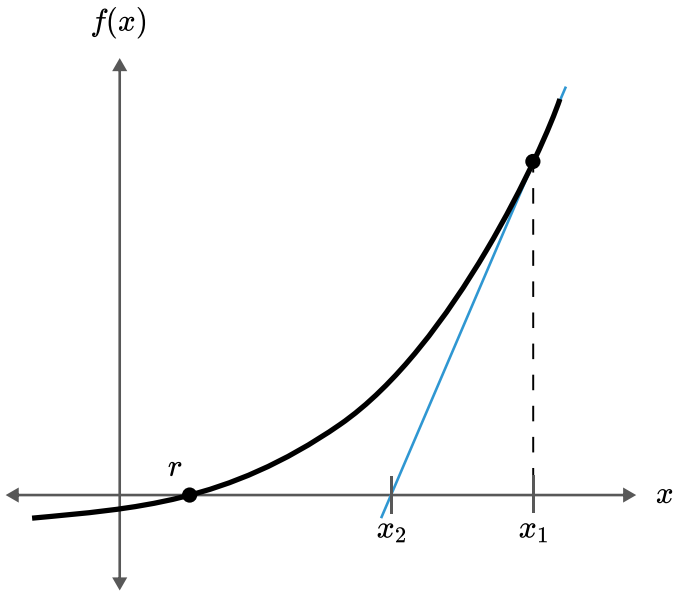

Figure 4.24

In the figure above, the root of the function $y = f(x)$ is r. In order to approximate the root r, we start with a value x_1 such that $x_1 \approx r$. Since $x_1 \neq r$ and $f(x_1) \neq 0$, therefore we draw a tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$. Now, observe that the tangent line is close to the graph of $f(x)$ near the point of tangency, therefore the graph of $f(x)$ and the tangent line at $(x_1, f(x_1))$ intercepts the x-axis at approximately the same point. This

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implies that the root r of $f(x)$ is approximately equal to the root of the tangent line, that is, x_2. Now, since the tangent line is a line and its root x_2 can easily be determined, therefore x_2 can be used as a second approximation to r (where, the first approximation was x_1) to repeat the process of approximation with a better precision.

Since the slope of the tangent line at the point x_1 is $f'(x_1)$, therefore the equation of the tangent line yields,

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Now, since $y = 0$ when $x = x_2$ (as x_2 lies on the x-axis) in the equation above, therefore,

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ 0 - f(x_1) &= f'(x_1)(x_2 - x_1) \\ -\frac{f(x_1)}{f'(x_1)} &= (x_2 - x_1) \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \end{aligned}$$

where, $f'(x) \neq 0$. Now, since $f(x_2) \neq 0$, therefore $x_2 \neq r$, and thus $x_2 \approx r$. A better approximation x_3 can now be derived by repeating the process disussed above with an initial approximation being x_2 to the root r. That is, draw a tangent line to the graph of $f(x)$ at the point $(x_2, f(x_2))$, that intercepts the x-axis at the point x_3. Therefore, the equation of the tangent line is as follows.

$$y - f(x_2) = f'(x_2)(x - x_2)$$

Since the tangent line intercepts the x-axis at the point $x = x_3$, therefore $y = 0$ when $x = x_3$. This implies,

$$\begin{aligned} y - f(x_2) &= f'(x_2)(x - x_2) \\ 0 - f(x_2) &= f'(x_2)(x_3 - x_2) \\ -\frac{f(x_2)}{f'(x_2)} &= x_3 - x_2 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \end{aligned}$$

where, $f'(x_2) \neq 0$. If $f(x_3) \neq 0$, then $x_3 \neq r$, and thus $x_3 \approx r$. This implies that the process can be repeated again by considering x_3 as the initial approximation. This yields a sequence of approximation $x_1, x_2, x_3, x_4, \cdots$, as shown in the following figure.

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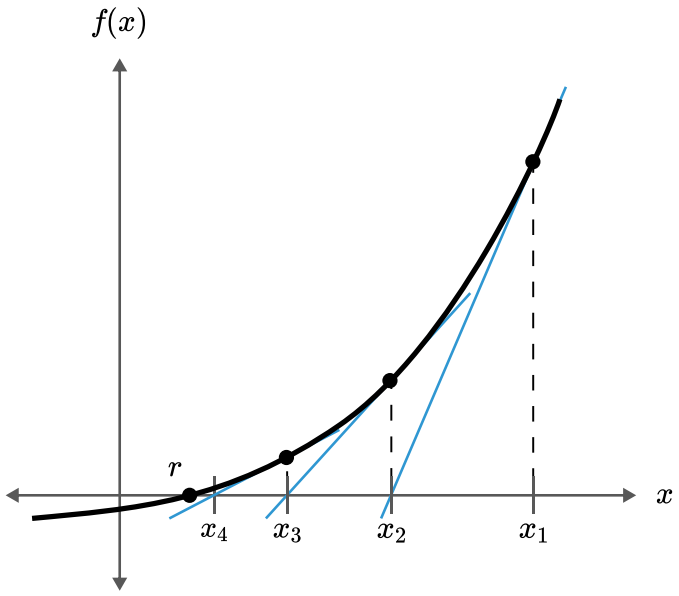

Figure 4.25

In general, if the nth approximation is $f(x_n)$ and $f'(x_n) \neq 0$, then the next approximation yields,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This implies that if $x_n \to r$, as $n \to \infty$, then the sequence of successive approximations converges to the actual root r of the function $f(x)$.

► **Example 4.7.1.** Approximate a root of $f(x) = x^4 - \pi$ using Newton's Method up to 7 iterations, where the initial approximation $x_1 = 1$.

Solution. Since $f(x) = x^4 - \pi$, therefore, $f'(x) = 4x^3$, and thus, by the Newton's Method,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^4 - \pi}{4x_n^3} \end{aligned}$$

This implies that if x_n is the initial approximation of a root r of $f(x) = x^4 - \pi$, then $x_{n+1} = x_n - \dfrac{x_n^4 - \pi}{4x_n^3}$ is its next better approximation. Below the calculations of these iterations are presented for $n = 1, 2, \cdots, 7$.

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n	x_n	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1	1.5353981634
2	1.5353981634	1.36853218292
3	1.36853218292	1.33282459009
4	1.33282459009	1.33133785791
5	1.33133785791	1.33133536381
6	1.33133536381	1.3313353638
7	1.3313353638	1.3313353638

It is evident that as the number of iterations n increases, the sequence of approximation converges to a specific value, and that specific value is a root of the function $f(x) = x^4 - \pi$. ∎

► **Example 4.7.2.** Approximate a root of $f(x) = e^{ex} - e$, using Newton's Method up to 7 iterations, where the initial approximation is $x_1 = 0.5$.

Solution. Since $f(x) = e^{ex} - e$, therefore, $f'(x) = e^{ex+1}$, and thus, by the Newton's Method,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{e^{ex_n} - e}{e^{ex_n+1}} \end{aligned}$$

Calculations of the iterations are presented below for $n = 1, 2, \cdots, 7$.

n	x_n	$x_{n+1} = x_n - \dfrac{e^{ex_n} - e}{e^{ex_n+1}}$
1	0.5	0.389001924142
2	0.389001924142	0.368474393475
3	0.368474393475	0.367879922005
4	0.367879922005	0.367879441172
5	0.367879441172	0.367879441171
6	0.367879441171	0.367879441171
7	0.367879441171	0.367879441171

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Although Newton's Method is a powerful technique to find the root for finding the roots of a function, it sometimes fails to do so. For example, if the derivative $f'(x_n)$ in the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ turns out to be 0 for any n, then the formula involves division by zero, making it undefined altogether.

Similarly, Newton's Method can also fail for other reasons. It might miss the aimed root and instead converge to a different one, or not converge at all. For example, consider the following example.

► **Example 4.7.3.** Approximate a root of $f(x) = \sqrt[3]{x}$, using Newton's Method up to 7 iterations, where the initial approximation is $x_1 = 1$.

Solution. Since $f(x) = \sqrt[3]{x}$, therefore $f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$. Now, by the Newton's Method,

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{\sqrt[3]{x_n}}{\dfrac{1}{3x_n^{\frac{2}{3}}}} \\
&= x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} \\
&= x_n - \frac{1}{\frac{1}{3}} \cdot x_n^{\frac{1}{3}+\frac{2}{3}} \\
&= x_n - 3 \cdot x_n^{\frac{3}{3}} \\
&= x_n - 3x_n \\
&= -2x_n
\end{aligned}$$

Calculations of the iterations are presented below for $n = 1, 2, \cdots, 7$.

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n	x_n	$x_{n+1} = -2x_n$
1	1	-2
2	-2	4
3	4	-8
4	-8	16
5	16	-32
6	-32	64
7	64	-128

It is evident that the sequence of approximation does not converge to a specific value, therefore Newton's Method fails in this case. ∎

———————The End of Section 4.7
Newton's Method

§ 4.8 L'Hôpital's Rule

The limit of a function and its evaluation were studied in Chapter 1. However, the techniques presented there sometimes either become cumbersome or fall short of certain limits known as *indeterminate forms*. For example, consider the following limit of a function,

$$\lim_{x\to\infty} \frac{ax}{x}$$

As $x \to \infty$, the numerator ax, as well as the denominator x increases without a bound, resulting in an indeterminate form of $\frac{\infty}{\infty}$. Nevertheless, a simple simplification suggests that,

$$\begin{aligned}\lim_{x\to\infty} \frac{ax}{x} &= \lim_{x\to\infty} a \\ &= a\end{aligned}$$

Similarly, consider the limit below,

$$\lim_{x\to 0} \frac{ax}{x}$$

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This time, as $x \to 0$, the numerator ax and the denominator x approaches 0, resulting again in an indeterminate form of $\frac{0}{0}$. Despite that, a simple simplification again suggests the following,

$$\begin{aligned}\lim_{x\to 0} \frac{ax}{x} &= \lim_{x\to 0} a \\ &= a\end{aligned}$$

In general, limits that take the form of $\frac{\infty}{\infty}$ or $\frac{0}{0}$ are indeterminate, and can have any value. Therefore, these types of limits require a different approach when determining their values.

The method often used in such situation is known as L'Hôpital's Rule, and the idea behind it is quite simple: when both the numerator and denominator of a fraction approach either 0 or ∞ as x approaches a certain value, directly evaluating the limit leads to an indeterminate form. In such cases, examining the rates at which the numerator and denominator change (i.e., their derivatives) can provide insight into the function's behavior. For example, if both the numerator and denominator approaches infinity, we can't immediately determine the limit of the fraction. The outcome is ambiguous. However, if we somehow know that the numerator of the fraction increases at a rapid pace while the denominator increases slowly, then this information becomes critical, as it shows that the behavior of the numerator largely dictates the fraction's limit. Therefore, instead of directly evaluating the numerator and denominator, we analyze their respective rates of change (i.e., derivatives) to determine the overall limit of the fraction. In the following subsections, we go through these cases one by one.

Chap. 4 / Sec. 4.8 / Subsec. 4.8.1 : Indeterminate Form $\frac{0}{0}$

Consider a function $h(x) = \frac{f(x)}{g(x)}$. If $\lim_{x\to a} f(x) = 0$ as well as $\lim_{x\to a} g(x) = 0$, then the limit $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, which is the indeterminate form of $\frac{0}{0}$. In order to determine the actual value of such a limit of indeterminate form, the L'Hôpital's Rule is used stating the following.

Theorem 4.8.1. $\left(\text{L'Hôpital's Rule for } \frac{0}{0}\right)$

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Let $f(x)$ and $g(x)$ be differentiable functions over an open interval I containing a, except possibly at a and suppose $g'(x) \neq 0$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then L'Hôpital's Rule states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right side exists (or is $\pm\infty$). The rule is also applicable for the cases where $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Proof. We prove a special case of this theorem where we assume that $f(a) = g(a) = 0$ and $f'(x)$ and $g'(x)$ are continuous at a, and $g'(a) \neq 0$. Thus, we have,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

By the definition of a derivative, we get,

$$\begin{aligned}
\frac{f'(a)}{g'(a)} &= \frac{\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a}} \\
&= \lim_{x\to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}} \\
&= \lim_{x\to a} \left(\frac{f(x) - f(a)}{\cancel{x - a}} \cdot \frac{\cancel{x - a}}{g(x) - g(a)} \right) \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}
\end{aligned}$$

Since, $f(a) = g(a) = 0$, therefore,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} &= \lim_{x\to a} \frac{f(x) - 0}{g(x) - 0} \\
&= \lim_{x\to a} \frac{f(x)}{g(x)}
\end{aligned}$$

This completes the proof. ∎

A few examples to demonstrate the theorem above are provided below.

► **Example 4.8.1.** Evaluate the limit $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

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Solution. Substituting $x = 0$ directly into the expression yields an indeterminate form of $\frac{0}{0}$,

$$\begin{aligned}\frac{e^x - 1}{x} &= \frac{e^0 - 1}{0} \\ &= \frac{0}{0}\end{aligned}$$

However, by the L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{dx}{dx}} \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}e^x - \frac{d}{dx}(1)}{\frac{dx}{dx}} \\ &= \lim_{x\to 0} \frac{e^x}{1} \\ &= e^0 \\ &= 1\end{aligned}$$

Therefore, $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. ∎

Remark 4.8.1.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

► **Example 4.8.2.** Evaluate the limit $\lim_{x\to 0} \frac{a^x - 1}{x}$

Solution. Substituting $x = 0$ directly into the expression yields an indeterminate form of $\frac{0}{0}$,

$$\begin{aligned}\frac{a^x - 1}{x} &= \frac{a^0 - 1}{0} \\ &= \frac{0}{0}\end{aligned}$$

However, by the L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(a^x - 1)}{\frac{dx}{dx}}$$

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$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\dfrac{d}{dx}a^x - \dfrac{d}{dx}(1)}{\dfrac{dx}{dx}} \\
&= \lim_{x\to 0} \frac{a^x \ln(a)}{1} \\
&= a^0 \ln(a) \\
&= \ln(a)
\end{aligned}
$$

Therefore, $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln(a)$ ∎

Remark 4.8.2.

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$$

► **Example 4.8.3.** Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$

Solution. Substituting $x = 0$ directly into the expression yields an indeterminate form of $\dfrac{0}{0}$,

$$
\begin{aligned}
\frac{\ln(1+x)}{x} &= \frac{\ln(1+0)}{0} \\
&= \frac{0}{0}
\end{aligned}
$$

However, by the L'Hôpital's Rule,

$$
\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+x)}{x} &= \lim_{x\to 0} \frac{\dfrac{d}{dx}\ln(1+x)}{\dfrac{dx}{dx}} \\
&= \lim_{x\to 0} \frac{\dfrac{1}{1+x}\cdot\dfrac{d}{dx}(1+x)}{\dfrac{dx}{dx}} \\
&= \lim_{x\to 0} \frac{\dfrac{1}{1+x}}{1} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}
$$

Therefore, $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$ ∎

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Remark 4.8.3.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

Chap. 4 / Sec. 4.8 / Subsec. 4.8.2 : Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule is also applicable to the limits of the indeterminant form $\frac{\infty}{\infty}$. For example, consider a function $h(x)=\frac{f(x)}{g(x)}$. If $\lim_{x\to a} f(x)=\pm\infty$ as well as $\lim_{x\to a} g(x)=\pm\infty$, then the limit $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$, which is the indeterminate form of $\frac{\infty}{\infty}$. Now to solve such a limit, L'Hôpital's Rule states the following.

Theorem 4.8.2. (L'Hôpital's Rule for $\frac{\pm\infty}{\pm\infty}$)

Let $f(x)$ and $g(x)$ be differentiable functions over an open interval I containing a, except possibly at a and suppose $g'(x)\neq 0$. If $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$, then L'Hôpital's Rule states that

$$\lim_{x\to a}\frac{f\left(x\right)}{g\left(x\right)}=\lim_{x\to a}\frac{f'\left(x\right)}{g'\left(x\right)}$$

provided that the limit on the right side exists (or is $\pm\infty$). The rule is also applicable for the cases where $x\to\pm\infty$, $x\to a^{+}$, or $x\to a^{-}$.

A few examples to demonstrate the theorem above are presented below.

► **Example 4.8.4.** Evaluate the limit $\lim_{x\to\infty}\frac{2x^2+3x+5}{2x^2-3x-5}$

Solution. As $x\to\infty$, the expression yields an indeterminate form of $\frac{\infty}{\infty}$, therefore, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^2+3x+5}{2x^2-3x-5}&=\lim_{x\to\infty}\frac{\frac{d}{dx}\left(2x^2+3x+5\right)}{\frac{d}{dx}\left(2x^2-3x-5\right)}\\&=\lim_{x\to\infty}\frac{4x+3}{4x-3}\end{aligned}$$

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Now, as $x \to \infty$, the expression again yields an indeterminate form of $\frac{\infty}{\infty}$, therefore, we reapply L'Hôpital's Rule once more,

$$\begin{aligned}\lim_{x\to\infty} \frac{4x+3}{4x-3} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(4x+3)}{\frac{d}{dx}(4x-3)} \\ &= \lim_{x\to\infty} \frac{4}{4} \\ &= 1\end{aligned}$$

Therefore, $\lim_{x\to\infty} \frac{2x^2+3x+5}{2x^2-3x-5} = 1$ ∎

► **Example 4.8.5.** Evaluate $\lim_{x\to\infty} \frac{\ln(1+x)}{x}$

Solution. As $x \to \infty$, the expression yields an indeterminate form of $\frac{\infty}{\infty}$, therefore, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(1+x)}{x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(1+x)}{\frac{dx}{dx}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)}{\frac{dx}{dx}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} \\ &= \lim_{x\to\infty} \frac{1}{1+x} \\ &= 0\end{aligned}$$

Therefore, $\lim_{x\to\infty} \frac{\ln(1+x)}{x} = 0$ ∎

Remark 4.8.4.

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = 0$$

► **Example 4.8.6.** Evaluate $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x-1)}$

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Solution. As $x \to \infty$, the expression yields an indeterminate form of $\frac{\infty}{\infty}$, therefore, by L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x-1)} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}\ln(x-1)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x+1}\cdot\frac{d}{dx}(x+1)}{\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x-1}} \\
&= \lim_{x\to\infty} \frac{x-1}{x+1}
\end{aligned}$$

Now, as $x \to \infty$, the expression again yields an indeterminate form of $\frac{\infty}{\infty}$, therefore, we reapply L'Hôpital's Rule once more,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x-1}{x+1} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x+1)} \\
&= \lim_{x\to\infty} \frac{1}{1} \\
&= 1
\end{aligned}$$

Therefore, $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x-1)} = 1$ ∎

Chap. 4 / Sec. 4.8 / Subsec. 4.8.3 : Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

In addition to the limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, there are other types of indeterminate forms. For example, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then the limit of a product of two functions $\lim_{x\to a} f(x)\,g(x)$ results in the indeterminate form $0\cdot\infty$. Unlike the previously mentioned indeterminate forms, L'Hôpital's Rule cannot be directly applied to these limits. To address them, they must first be converted into limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as shown in the examples below.

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► **Example 4.8.7.** Evaluate $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right)$

Solution. Since the limit $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right)$ is of the indeterminate form $0\cdot\infty$, therefore L'Hôpital's Rule cannot be directly applied to it. As a result, this limit needs some modification to convert the form into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. This can be done by the use of reciprocals. For example, since the reciprocal of x^2 is $\frac{1}{x^2}$, therefore $x^2 = \frac{1}{\frac{1}{x^2}}$. As a result, the limit can be rewritten as follows,

$$\begin{aligned}\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}} \sin\left(\frac{1}{x}\right)\\ &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x^2}}\end{aligned}$$

Now, as $x\to\infty$, both the numerator and denominator approach 0, thus the limit is converted into the indeterminate form $\frac{0}{0}$ such that L'Hôpital's Rule now can be directly applied to evaluate the limit. That is,

$$\begin{aligned}\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x^2}} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\sin\left(\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x^2}\right)}\\ &= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)}{-\frac{2}{x^3}}\\ &= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}\\ &= \lim_{x\to\infty} \frac{-\frac{\cos\left(\frac{1}{x}\right)}{x^2}}{-\frac{2}{x^3}}\end{aligned}$$

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$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)}{x^2} \cdot \frac{x^3}{2} \\
&= \lim_{x\to\infty} \frac{x\cos\left(\frac{1}{x}\right)}{2} \\
&= \frac{\infty \cdot \overbrace{\cos\left(\frac{1}{\infty}\right)}^{1}}{2} \\
&= \infty
\end{aligned}
$$

This implies that $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right) = \infty$. ∎

Another type of indeterminate form is $\infty - \infty$. For example, if $\lim_{x\to a} f(x) = \infty$ and if $\lim_{x\to a} g(x) = \infty$, then the limit of the function $\lim_{x\to a}\Big(f(x) - g(x)\Big)$ yields the indeterminate form $\infty - \infty$. Contrary to the previous indeterminate forms, L'Hôpital's Rule cannot be directly applied to these limits and to address them, they must first need to be converted into limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

► **Example 4.8.8.** Evaluate $\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$

Solution. As $x \to 0$, both terms $\cot(x)$ and $\frac{1}{x}$ approach ∞, yielding an indeterminate form $\infty - \infty$. As a result, we do the following simplification to convert into limits of the indeterminate form $\frac{\infty}{\infty}$.

$$
\begin{aligned}
\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)} - \frac{1}{x}\right) \\
&= \lim_{x\to 0}\left(\frac{x\cos(x) - \sin(x)}{x\sin(x)}\right)
\end{aligned}
$$

Now, we apply L'Hôpital's Rule,

$$
\begin{aligned}
\lim_{x\to 0} \frac{\frac{d}{dx}\left(x\cos(x) - \sin(x)\right)}{\frac{d}{dx}\left(x\sin(x)\right)} &= \lim_{x\to 0} \frac{\frac{d}{dx}\left(x\cos(x)\right) - \frac{d}{dx}\sin(x)}{\frac{d}{dx}\left(x\sin(x)\right)} \\
&= \lim_{x\to 0} \frac{x\cdot\frac{d}{dx}\cos(x) + \cos(x)\cdot\frac{dx}{dx} - \cos(x)}{x\cdot\frac{d}{dx}\sin(x) + \sin(x)\cdot\frac{dx}{dx}}
\end{aligned}
$$

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$$= \lim_{x\to 0} \frac{-x\sin(x) + \cancel{\cos(x)} - \cancel{\cos(x)}}{x\cos(x) + \sin(x)}$$
$$= -\lim_{x\to 0} \frac{x\sin(x)}{x\cos(x) + \sin(x)}$$

Reapplying L'Hôpital's Rule again yields,

$$\begin{aligned}
-\lim_{x\to 0} \frac{x\sin(x)}{x\cos(x) + \sin(x)} &= -\lim_{x\to 0} \frac{\frac{d}{dx}(x\sin(x))}{\frac{d}{dx}(x\cos(x) + \sin(x))} \\
&= -\lim_{x\to 0} \frac{\frac{d}{dx}(x\sin(x))}{\frac{d}{dx}(x\cos(x)) + \frac{d}{dx}\sin(x)} \\
&= -\lim_{x\to 0} \frac{x\cdot\frac{d}{dx}\sin(x) + \sin(x)\cdot\frac{dx}{dx}}{x\cdot\frac{d}{dx}\cos(x) + \cos(x)\cdot\frac{dx}{dx} + \cos(x)} \\
&= -\lim_{x\to 0} \frac{x\cos(x) + \sin(x)}{-x\sin(x) + \cos(x) + \cos(x)} \\
&= -\frac{0\cdot\cos(0) + \sin(0)}{-x\sin(0) + \cos(0) + \cos(0)} \\
&= -\frac{0}{1+1} \\
&= 0
\end{aligned}$$

Therefore, $\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right) = 0$ ∎

It is sometimes helpful to change the variable. That is, to evaluate $\lim_{x\to\infty} f(x)$, it is sometimes helpful to let $t = \frac{1}{x}$. Now, notice that if $x \to \infty$, then $t \to 0^+$. As a result, we can rewrite the limit in terms of t, as follows,

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right)$$

Note that since we let $t = \frac{1}{x}$, therefore $x = \frac{1}{t}$. Thus, if $x \to +\infty$, then $t \to 0$ from the positive side.

► **Example 4.8.9.** Evaluate $\lim_{x\to\infty}\left(x - \sqrt{x^2+6x}\right)$

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Solution. As $x \to \infty$, both the terms x as well as $\sqrt{x^2+6x}$ approach ∞, yielding an indeterminate form $\infty - \infty$. Nevertheless, to evaluate this limit, we do the following,

$$\begin{aligned}\lim_{x\to\infty}\left(x-\sqrt{x^2+6x}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{6x}{x^2}\right)}\right)\\ &= \lim_{x\to\infty}\left(x-\sqrt{x^2}\cdot\sqrt{1+\frac{6}{x}}\right)\\ &= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{6}{x}}\right)\end{aligned}$$

Now, let a new variable $t = \dfrac{1}{x}$, then $x = \dfrac{1}{t}$. Substituting these variables yields,

$$\begin{aligned}\lim_{x\to\infty} x\left(1-\sqrt{1+\frac{6}{x}}\right) &= \lim_{t\to 0^+}\frac{1}{t}\left(1-\sqrt{1+6t}\right)\\ &= \lim_{t\to 0^+}\frac{1-\sqrt{1+6t}}{t}\end{aligned}$$

Now, as $t \to 0^+$, both the numerator and denominator yield an indeterminate form $\dfrac{0}{0}$, and L'Hôpital's Rule can now be directly applied. L'Hôpital's Rule states that,

$$\begin{aligned}\lim_{t\to 0^+}\frac{1-\sqrt{1+6t}}{t} &= \lim_{t\to 0^+}\frac{\dfrac{d}{dt}\left(1-\sqrt{1+6t}\right)}{\dfrac{d}{dt}t}\\ &= \lim_{t\to 0^+}\frac{-\dfrac{1}{2\sqrt{1+6t}}\cdot\dfrac{d}{dt}(1+6t)}{1}\\ &= \lim_{t\to 0^+}\frac{-\dfrac{1}{2\sqrt{1+6t}}\cdot 6}{1}\\ &= \lim_{t\to 0^+}-\frac{6}{2\sqrt{1+6t}}\\ &= -\frac{6}{2\sqrt{1+0}}\\ &= -\frac{6}{2}\\ &= -3\end{aligned}$$

This implies that $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+6x}\right) = -3$ ∎

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Chap. 4 / Sec. 4.8 / Subsec. 4.8.4 : Indeterminate Forms 1^∞, 0^0, and ∞^0

There are other indeterminate forms 1^∞, 0^0, and ∞^0 that arise from the limit of the following form,

$$\lim_{x\to a} f(x)^{g(x)}$$

The indeterminate form 1^∞ arises if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$. The indeterminate form 0^0 arises if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Likewise, the indeterminate form ∞^0 arises if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.

Contrary to the previous forms, L'Hôpital's Rule cannot directly be applied to them. Therefore, in order to apply L'Hôpital's Rule, they must first need to be expressed in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by manipulation.

In order to convert them into the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the inverse relationship between $\ln(x)$ and e^x is used. That is, if $y = f(x)^{g(x)}$, then

$$\begin{aligned}
\ln(y) &= \ln\left(f(x)^{g(x)}\right)\\
&= g(x)\ln f(x)\\
e^{\ln(y)} &= e^{g(x)\ln f(x)}\\
y &= e^{g(x)\ln f(x)}\\
f(x)^{g(x)} &= e^{g(x)\ln f(x)}
\end{aligned}$$

This implies that we can rewrite the limit $\lim_{x\to a} f(x)^{g(x)}$ as,

$$\begin{aligned}
\lim_{x\to a} f(x)^{g(x)} &= \lim_{x\to a} e^{g(x)\ln f(x)}\\
&= e^{\lim_{x\to a} g(x)\ln f(x)}
\end{aligned}$$

provided that the limit $\lim_{x\to a} g(x)\ln f(x)$ exists.

► **Example 4.8.10.** Evaluate $\lim_{x\to 0^+} (\sin(x))^{\tan(x)}$

Solution. As $x \to 0$ from the positive side, $\sin(x)^{\tan(x)} \to 0^0$. This implies that the limit has the indeterminate form 0^0. However, notice that,

$$\ln(\sin(x))^{\tan(x)} = \tan(x)\ln(\sin(x))$$

This implies that

$$\lim_{x\to 0^+} (\sin(x))^{\tan(x)} = \lim_{x\to 0^+} \tan(x)\ln(\sin(x))$$

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Now, we convert this to a limit of the indeterminate form $\frac{0}{0}$ and apply L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to 0^+} \tan(x)\ln(\sin(x)) &= \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\cot(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(\sin(x))}{\frac{d}{dx}\cot(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{\sin(x)}\frac{d}{dx}\sin(x)}{-\csc^2(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{\cos(x)}{\sin(x)}}{-\frac{1}{\sin^2(x)}} \\
&= -\lim_{x\to 0^+} \frac{\cos(x)}{\sin(x)}\cdot\sin^2(x) \\
&= -\lim_{x\to 0^+} \cos(x)\sin(x) \\
&= -\cos(0)\sin(0) \\
&= 0
\end{aligned}$$

Therefore, $\lim_{x\to 0^+} (\sin(x))^{\tan(x)} = 0$ ∎

► **Example 4.8.11.** Evaluate $\lim_{x\to 0^+} x^x$.

Solution. As $x \to 0$ from the positive side, $x^x \to 0^0$. This implies that the limit has the indeterminate form 0^0. To evaluate the limit we let, $x^x = e^{x\ln(x)} = e^L$ and solve the limit $L = \lim_{x\to 0^+} x\ln(x)$. The limit $\lim_{x\to 0^+} x\ln(x)$ has an indeterminate form $0\cdot\infty$, therefore to use L'Hôpital's Rule, the indeterminate form $0\cdot\infty$ is modified into the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as shown below.

$$\begin{aligned}
\lim_{x\to 0^+} x\ln(x) &= \lim_{x\to 0^+} \frac{1}{\frac{1}{\ln(x)}}\cdot\ln(x) \\
&= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}
\end{aligned}$$

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Since the limit $\lim\limits_{x\to 0^+} \dfrac{\ln(x)}{\dfrac{1}{x}}$ has the indeterminate form $\dfrac{0}{0}$, therefore L'Hôpital's Rule is used stating that,

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} \frac{1}{x}\cdot\left(-x^2\right) \\
&= \lim_{x\to 0^+} -x \\
&= 0
\end{aligned}$$

This implies that $L = \lim\limits_{x\to 0^+} x\ln(x) = 0$. Since $L = \lim\limits_{x\to 0^+} x\ln(x) = 0$, therefore

$$\begin{aligned}
\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{x\ln(x)} \\
&= \lim_{x\to 0^+} e^0 \\
&= \lim_{x\to 0^+} 1 \\
&= 1
\end{aligned}$$

As a result, $\lim\limits_{x\to 0^+} x^x = 1$. ∎

———————The End of Section 4.8

L'Hôpital's Rule

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Part III

Integral Calculus

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Chapter 5

An Introduction to Integral Calculus

Integral Calculus is a subfield of calculus that studies the concept of integration—a process of finding the integral of a given function. An integral can most easily be understood as a way to accumulate quantities, such as the area of a region. The area of such a region represents the measure of space that a two-dimensional shape occupies.

Humans have long been familiar with the concept of area, even in ancient times, and have developed various methods to calculate them. For example, the area of a rectangle is determined by multiplying its base by its height. Similarly, the area of a triangle is calculated as half the area of a rectangle since a triangle can be thought of as one-half of a rectangle when divided by a diagonal. Likewise, the area of a regular polygon can be found by dividing the polygon into smaller triangles and summing their areas.

Contrary to the shapes above, the shape of a circle is complex for the fact that it has no corners or edges, resulting in a smooth, continuous, and round curve. Finding the area of a circle is thus generally more difficult than the ones mentioned above. Documented history suggests that it was the Greeks, particularly Archimedes, who were among the first to solve the problem of finding the area of a circle through approximation. The method used in that process is referred to as the "method of exhaustion" for the reason that it involves

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inscribing[1] (or circumscribing) a regular polygon into a circle and increasing its sides in a way that the area of the polygon approaches the exact area of the circle. For example, consider the circle below.

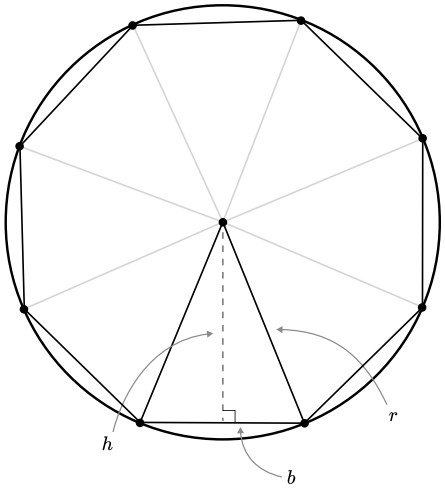

Figure 5.1

In the figure above, a regular polygon is inscribed into the circle. Since each of the isosceles triangles has an area $A = \frac{1}{2}hb$, therefore the sum of these areas equals the area of the polygon, which is a close approximation of the area of the circle. That is, the area of the polygon A_p yields,

$$\begin{aligned} A_p &= \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb + \cdots + \frac{1}{2}hb \\ &= \frac{1}{2}h\left(b + b + b + \cdots + b\right) \\ &= \frac{1}{2}hp \end{aligned}$$

where, p is the perimeter of the polygon. Now, if $C = 2\pi r$ is the circumference of the circle, then as the number of sides in the polygon increases, the height h in each polygon approaches the circle's radius r, that is, $h \to r$; and the perimeter of the polygon approaches the circumference C, that is, $p \to C$. This implies that,

$$A_p = \frac{1}{2}hp \to \frac{1}{2}rC$$

[1] to fit a smaller shape inside a larger shape in such a way that the smaller shape touches but does not intersect the larger shape.

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$$= \frac{1}{2} r \left(2\pi r\right)$$
$$\therefore A_p \to \pi r^2$$

As a result, the area A of a circle with radius r is found to be,

$$A = \pi r^2$$

as the sides of the polygon increase indefinitely.

The method of exhaustion works effectively for circles due to their regular geometric properties. However, when applied to irregular shapes lacking such properties, the method loses its value—it becomes impractical. Therefore, we need to consider a different approach to calculate the area of irregular shapes. In modern terms, the problem can be generalized to finding the area of a region with a curved boundary.

To address this, we begin with a specific case of the general problem: find the area under the graph of a function $y = f(x)$ and above the x-axis, between two vertical lines at $x = a$ and $x = b$. This enclosed region has a curved boundary only along its upper edge, making it easier to handle. This generalized problem is commonly referred to as finding the "area under a curve."

Solving this specific case provides a foundation for addressing more complex regions, such as those with curved boundaries on both edges. For such regions, the total area can be determined by subtracting the area under the lower curve from the area under the upper curve, where both areas belong to the simpler "area under a curve" category. These principles are discussed further in the following.

§ 5.1 The Area Under a Curve

The term "area under a curve" refers to the total area enclosed by the graph of a function $f(x)$ and the x-axis, between two vertical lines from $x = a$ to $x = b$ (i.e., on the interval $[a, b]$). For example, consider the figure of the function $f(x) = \dfrac{x^2}{4}$ below.

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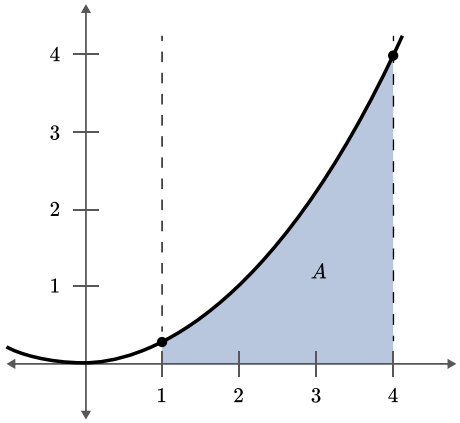

Figure 5.2

Suppose, we want to determine the exact area A under the curve of $f(x) = \frac{x^2}{4}$ and above the x-axis, from $x = 1$ to $x = 4$. In order to do that, we begin by breaking down the area into simple, familiar shapes—rectangles.

The idea is straightforward: the area under the curve can be approximated by adding up the areas of infinitely many narrow rectangles, where each rectangle's height is determined by the function's value at different points. As we increase the number of rectangles indefinitely, the approximation becomes more accurate, eventually giving us the exact area.

However, since we're dealing with infinitely many rectangles, we need a systematic way to represent their sum. That is, summing these infinitely many rectangles will be impossible for us to do in the traditional way, so we need some sort of mathematical "tool" to handle this situation more compactly. And this is where the summation notation, $\sum$, comes into play to help us neatly organize and compute the total area. We discuss it in the following subsection very shortly and then move to calculate the area under the curve.

Chap. 5 / Sec. 5.1 / Subsec. 5.1.1 : The Summation ($\sum$) Notation

The summation notation, also known as the sigma notation, is a concise way to represent a large sum of numbers that follow a specific pattern. For example, if we ever need to calculate the sum of integers from 1 to 100, how would we express it? The common approach is to place the '+' sign in between each

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integer and to use '$\cdots$' to indicate skipping some of the terms as follows,

$$1 + 2 + 3 + 4 + 5 + \cdots + 96 + 97 + 98 + 99 + 100$$

However, that too isn't systematically concise enough compared to just explicitly expressing,

"The sum of integers i, from $i = 1$ to $i = 100$"

As a result, the summation notation was developed with the purpose to address situations like the one above. In that notation, we write,

$$\sum_{i=1}^{100} i$$

to mean the sum of all i such that i ranges from 1 to 100. In other words, it is the sum of all integers i from $i = 1$ to $i = 100$.

Here, i is called the index of summation. It may be obvious that the choice of the variable for the index does not matter at all as long as it doesn't confuse with the other variables that are in use. There can be any variable in the place of i to keep track of the terms to be added. For example, $\sum_{i=1}^{100} i = \sum_{k=1}^{100} k = \sum_{j=1}^{100} j$ conveys the same meaning. In fact, the index variable is called a *dummy variable* as it has generally no effect upon its choice. It is, however, customary to use the letters i, j, k for indices.

So, in general, if we want to take the sum of a set of real numbers $a_1, a_2, a_3, \cdots, a_n$, we write it as follows,

$$a_1 + a_2 + a_3 + \cdots + a_n$$

. Which can be written more compactly using the summation notation as follows,

$$\sum_{i=1}^{n} a_i$$

where i is the index of summation, a_i is the ith term of the summation, and the start and end limits of the summation are 1 and n, respectively. Therefore,

Remark 5.1.1. (The Summation Notation)

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The sum

$$a_1 + a_2 + a_3 + \cdots + a_n$$

can be concisely expressed as

$$\sum_{i=1}^{n} a_i$$

A few examples are presented below to demonstrate the use of the summation notation discussed above.

► **Example 5.1.1.** Express $2 + 5 + 8 + 11 + \cdots + 29$ in the summation notation.

Solution. This is an arithmetic sequence where the first term is 2, and the common difference is 3. Therefore,

$$\sum_{i=1}^{10} (2 + 3(i-1))$$

Therefore, $\displaystyle\sum_{i=1}^{10} (2 + 3(i-1)) = 2 + 5 + 8 + 11 + \cdots + 29.$ ∎

► **Example 5.1.2.** Express $3 + 6 + 12 + 24 + 48$ in the summation notation.

Solution. This is a geometric sequence with the first term $a = 3$ and common ratio $r = 2$. Therefore,

$$\sum_{i=0}^{4} 3 \cdot 2^i$$

Therefore, $\displaystyle\sum_{i=0}^{4} 3 \cdot 2^i = 3 + 6 + 12 + 24 + 48$ ∎

► **Example 5.1.3.** Express $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16}$ in terms of the summation notation.

Solution. In order to use the summation notation, notice the denominator of each term is a square of numbers from 1 to 4. That is,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$$

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and there are a total of 4 terms. Thus, the sum can be expressed as,

$$\sum_{k=1}^{4} \frac{1}{k^2}$$

.

Therefore, $\sum_{k=1}^{4} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$. ∎

A few summation rules are provided below to simplify the process for large and complex summation.

Rule 5.1.1 (Sigma Notation Rules). The following rules demonstrate the linearity of Sigma notation.

1. Constant Multiple Rule: $\sum_{i=1}^{n} ca_i = c \times \sum_{i=1}^{n} a_i$
2. Summation Rule: $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. Difference Rule: $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

A few examples are shown below to demonstrate the rules above.

► **Example 5.1.4.** Evaluate $\sum_{i=1}^{4} (i + 2i)$

Solution. Using the summation rule, we can split the sum into two separate sums,

$$\begin{aligned}\sum_{i=1}^{4} (i + 2i) &= \sum_{i=1}^{4} i + \sum_{i=1}^{4} 2i \\ &= \sum_{i=1}^{4} i + 2\sum_{i=1}^{4} i\end{aligned}$$

Now, we calculate the sum $\sum_{i=1}^{4} i$,

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4$$

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$$= 10$$

This implies that,

$$\begin{aligned}\sum_{i=1}^{4}(i+2i) &= \sum_{i=1}^{4} i + 2\sum_{i=1}^{4} i \\ &= 10 + 2 \cdot 10 \\ &= 30\end{aligned}$$

▮

► **Example 5.1.5.** Evaluate $\sum_{i=1}^{3}\left(2i - i^2\right)$

Solution. Using the summation rule, we can split the sum into two separate sums,

$$\sum_{i=1}^{3}\left(2i - i^2\right) = \sum_{i=1}^{3} 2i - \sum_{i=1}^{3} i^2$$

Now, we calculate each sum individually,

$$\begin{aligned}\sum_{i=1}^{3} 2i &= 2\sum_{i=1}^{3} i \\ &= 2\left(1+2+3\right) \\ &= 12\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{3} i^2 &= 1^2 + 2^2 + 3^2 \\ &= 1 + 4 + 9 \\ &= 14\end{aligned}$$

This implies that,

$$\begin{aligned}\sum_{i=1}^{3}\left(2i - i^2\right) &= \sum_{i=1}^{3} 2i - \sum_{i=1}^{3} i^2 \\ &= 12 - 14 \\ &= -2\end{aligned}$$

▮

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So far, we have calculated the sum manually. For example, to solve the sum $\sum_{i=1}^{n} i$, where $n = 3$, we would manually calculate it as $1 + 2 + 3 = 6$. However, what happens when the sum is to be calculated for $n = 1000, 10000, 100000$, etc.? Clearly, manual calculation falls short in such situations. Therefore, we have the following formulas.

Theorem 5.1.1. Summation formulas are provided below,

1. $\sum_{i=1}^{n} c = n \times c$
2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

A few examples are provided below to demonstrate the formulas above.

► **Example 5.1.6.** Calculate $1 + 2 + 3 + \cdots + 10000$.

Solution. From the formula above,

$$\begin{aligned}\sum_{i=1}^{10000} i &= \frac{10000(10000+1)}{2} \\ &= \frac{100010000}{2} \\ &= 50005000\end{aligned}$$

Therefore, $1 + 2 + 3 + \cdots + 10000 = 50005000$ ∎

► **Example 5.1.7.** Calculate $1^2 + 2^2 + 3^2 + \cdots + 10000^2$.

Solution. From the formula above,

$$\sum_{i=1}^{100} i^2 = \frac{10000(10000+1)(2 \cdot 10000+1)}{6}$$

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$$= \frac{2000300010000}{6}$$
$$= 333383335000$$

Therefore, $1^2 + 2^2 + 3^2 + \cdots + 10000^2 = 333383335000$ ∎

Having discussed the summation notation as a sort of prerequisite to start the discussion of the area under a curve, it is now the time that we may finally discuss that topic. We start off by discussing the area approximation in the following subsection.

Chap. 5 / Sec. 5.1 / Subsec. 5.1.2 : Area Approximation

With an understanding of the summation notation, we now calculate the exact area A under the curve of $f(x) = \frac{x^2}{4}$ and above the x-axis from $x = 1$ to $x = 4$, as shown below.

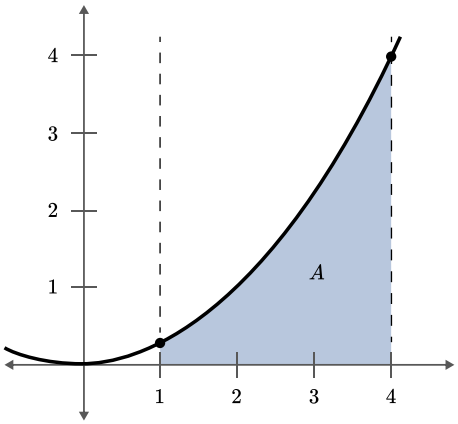

Figure 5.3

There are no simple formulas to calculate the area of this particular shape. However, its area can be approximated to any desired degree of accuracy by using the areas of familiar shapes, such as rectangles. The idea here is that the area A can be bounded between a lower area limit L and an upper area limit U, that is, $L \leq A \leq U$ (see Figure 5.4), in a way that gives a relatively good approximation of the exact area A in terms of the lower and upper area limit L and U, respectively for the reason that L and U are easy to calculate.

Now, to calculate the values of these lower and upper area limits L and U, we take a visual approach by considering the figure below.

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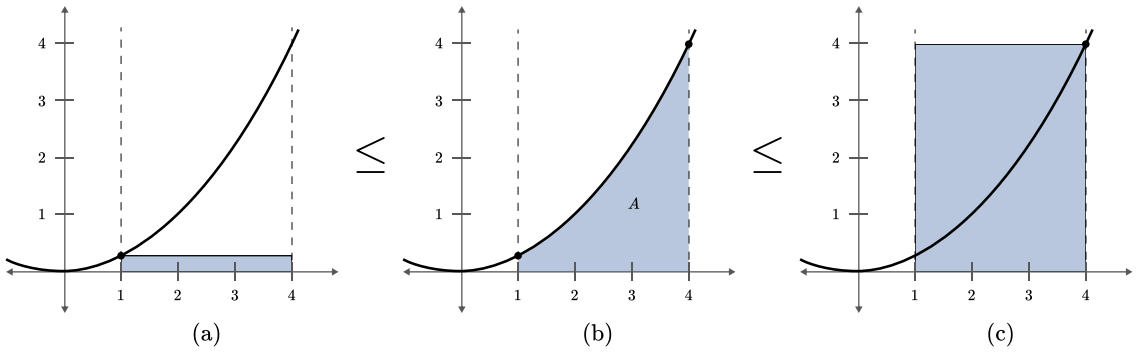

Figure 5.4

The lower area limit L of the exact area A is shown in Figure 5.4(a), and the upper area limit U is in Figure 5.4(c).

The area of the rectangle shown in Figure 5.4(a) is 0.75. This is because the width of the rectangle is $4 - 1 = 3$ and its height is the value of the function $f(x) = \dfrac{x^2}{4}$ at the point $x = 1$, that is, $f(1) = \dfrac{1^2}{4} = \dfrac{1}{4}$, thus, its area yields $\dfrac{1}{4} \times 3 = 0.75$.

Similarly, the area of the rectangle shown in Figure 5.4(c) yields 12. This is because the width of the rectangle is $4-1 = 3$ and its height is $f(4) = \dfrac{4^2}{4} = \dfrac{16}{4} = 4$, resulting the area to be $4 \times 3 = 12$.

Notice that the shaded region shown in Figure 5.4(a) is completely contained within Figure 5.4(b), but not all of the shaded region in Figure 5.4(b) is within Figure 5.4(a). Therefore, the area of the shaded region in Figure 5.4(a) is strictly smaller than that of Figure 5.4(b), thus $\boxed{0.75 < A}$.

Similarly, notice that the shaded region shown in Figure 5.4(b) is completely contained within Figure 5.4(c), but not all of the shaded region in Figure 5.4(c) is within Figure 5.4(b). Therefore, the area of the shaded region in Figure 5.4(b) is strictly smaller than that of Figure 5.4(c), thus $\boxed{A < 12}$.

This implies that the area of the shaded region in Figure 5.4(b) is strictly less than that of Figure 5.4(c) and greater than that of Figure 5.4(a), therefore, the exact area A is bounded by the inequality $\boxed{0.75 < A < 12}$.

As a result, we've derived an approximation of the exact area A in the form of an

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inequality. That being so, it is important to highlight a few considerations. For example, notice that the interval on which we're approximating the exact area is $[1, 4]$. This means each rectangle's width is 3 units as $4 - 1 = 3$. The height of which is determined by the function's value at the left and right endpoints of the interval $[1, 4]$, respectively. In this way, we've constructed rectangles that are either exactly under the curve or exactly above the curve. That is, we've constructed rectangles such that they're either underestimating or overestimating the exact area A. This is important to consider as choosing endpoints other than the left or right of the interval $[1, 4]$ would lead to rectangles that are neither exactly under the curve nor exactly over the curve, making the inequality quite ambiguous, as shown below.

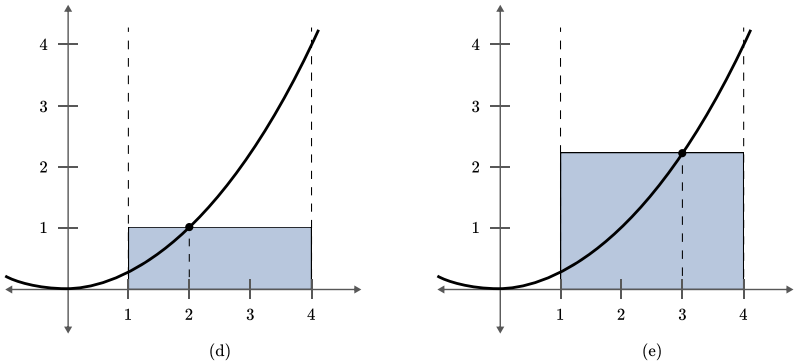

Figure 5.5

Now, turning back to our approximation, we see that although the range $0.75 < A < 12$ provides a decent estimation, we can achieve a more accurate approximation of the exact area A by using a greater number of rectangles. Previously, we used only one rectangle, but by increasing the number of rectangles, we can get a better approximation for the reason described below using the following figure.

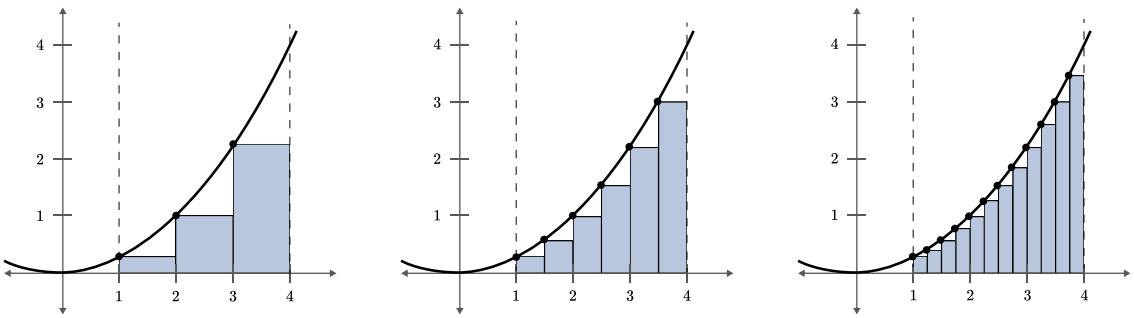

Figure 5.6

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Intuitively, it is clear that when we consider more rectangles, each rectangle becomes thinner to fit within the given interval. As a result, these rectangles cover more and more of the exact area A under the graph of $f(x) = \frac{x^2}{4}$. This intuition can be shown to be mathematically valid by comparing the results obtained from constructing three rectangles—each under and over the curve—versus those obtained from a single rectangle.

For that, we need to first divide the interval $[1, 4]$ into sub-intervals for determining the left and right endpoints of each sub-intervals, as previously explained. Let n be the number of rectangles constructed, then if the width of the entire interval $[1, 4]$ is $4 - 1$, then constructing $n = 3$ rectangles would make the width of each rectangle equal to $\frac{4-1}{3} = \frac{3}{3} = 1$. The height of each rectangle would be the function's value at either endpoint of each sub-interval, as before.

With that, let us first underestimate the area A using the left-endpoint approximation with 3 rectangles as follows.

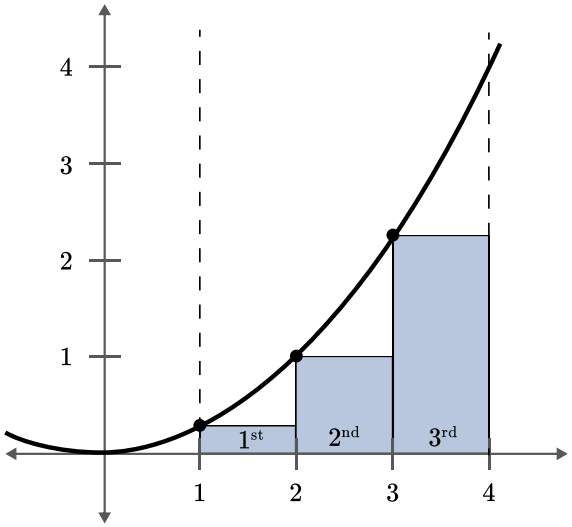

Figure 5.7

For the 1^{st} rectangle:

1. Height: $f(1) = \frac{1^2}{4} = 0.25$
2. Width: 1
3. Area_1: height $\times$ width $= 0.25 \times 1 = 0.25$

For the 2^{nd} rectangle:

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1. Height: $f(2) = \dfrac{2^2}{4} = \dfrac{4}{4} = 1$
2. Width: 1
3. Area_2: height $\times$ width $= 1 \times 1 = 1$

For the 3rd rectangle:

1. Height: $f(3) = \dfrac{3^2}{4} = \dfrac{9}{4} = 2.25$
2. Width: 1
3. Area_3: height $\times$ width $= 2.25 \times 1 = 2.25$

Total underestimation:

$$\begin{aligned}\text{Total Area} &= \text{Area}_1 + \text{Area}_2 + \text{Area}_3 \\ &= 0.25 + 1 + 2.25 \\ &= 3.5\end{aligned}$$

Now, we overestimate the exact area A using the left-endpoint approximation with 3 rectangles.

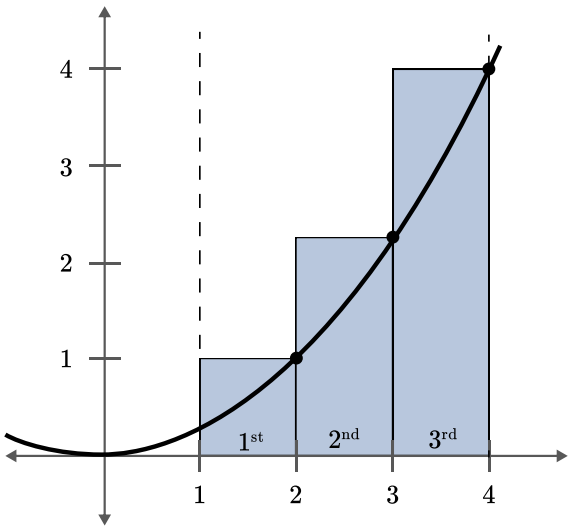

Figure 5.8

For the 1st rectangle:

1. Height: $f(2) = \dfrac{2^2}{4} = \dfrac{4}{4} = 1$

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2. Width: 1
3. Area_1: height $\times$ width $= 1 \times 1 = 1$

For the 2nd rectangle:

1. Height: $f(3) = \dfrac{3^2}{4} = \dfrac{9}{4} = 2.25$
2. Width: 1
3. Area_2: height $\times$ width $= 2.25 \times 1 = 2.25$

For the 3rd rectangle:

1. Height: $f(4) = \dfrac{4^2}{4} = \dfrac{16}{4} = 4$
2. Width: 1
3. Area_3: height $\times$ width $= 4 \times 1 = 4$

Total overestimation:

$$\begin{aligned} \text{Total Area} &= \text{Area}_1 + \text{Area}_2 + \text{Area}_3 \\ &= 1 + 2.25 + 4 \\ &= 7.25 \end{aligned}$$

This implies that using three rectangles to approximate the exact area A gives $\boxed{3.5 < A < 7.25}$, which is clearly a better approximation compared to the previous one, where the bound for the exact area A was $0.75 < A < 12$. Thus, it has been mathematically shown that using more rectangles provides a closer approximation of the exact area A.

Since more rectangles cover more of the area A, this means an even better approximation is always possible as there is no restriction to the number of rectangles that can be fitted within a given interval. However, manually calculating the height, width, and area of each rectangle, and then summing them up for the total estimation, is going to be quite tedious for large values. For example, if we fit a billion rectangles within a given interval, is it possible to manually calculate the area of all those rectangles a billion times? Therefore, let us now generalize the entire process for any arbitrary function $f(x)$, in the following subsection.

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Chap. 5 / Sec. 5.1 / Subsec. 5.1.3 : Generalized Area Approximation

Let $y = f(x)$ be any continuous, non-negative arbitrary function defined over the interval $[a, b]$ as shown below.

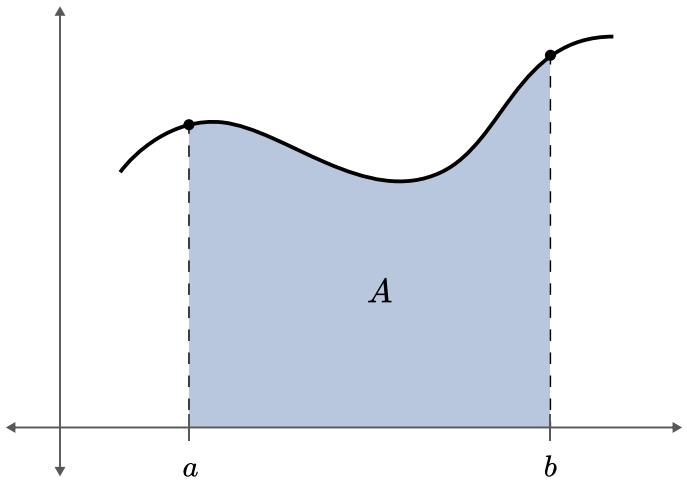

Figure 5.9

To determine the exact area A bounded under the curve of $f(x)$ and above the x-axis, from $x = a$ to $x = b$, we divide the interval $[a, b]$ into n sub-intervals of equal width. To do that, we use a set of increasing points $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ that are equally spaced between $x = a$ and $x = b$.

Since the total width of the interval $[a, b]$ is $b - a$, therefore dividing it into n parts gives each sub-interval a width of $\dfrac{b-a}{n}$. We denote this as Δx, thus $\Delta x = \dfrac{b-a}{n}$. Therefore, Δx represents the space between each of the points $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$, as shown in the figure below.

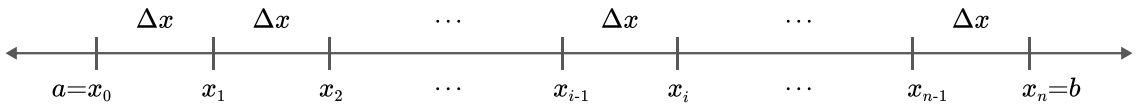

Figure 5.10

This implies that the interval $[a, b]$ is divided into the following n sub-intervals:

$$\underbrace{[x_0, x_1]}_{1^{\text{st}}}, \underbrace{[x_1, x_2]}_{2^{\text{nd}}}, \underbrace{[x_2, x_3]}_{3^{\text{rd}}}, \cdots, \underbrace{[x_{i-1}, x_i]}_{i^{\text{th}}}, \cdots, \underbrace{[x_{n-1}, x_n]}_{n^{\text{th}}}$$

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where,

$$\begin{aligned} x_1 &= x_0 + \Delta x \\ x_2 &= x_0 + 2\Delta x \\ x_3 &= x_0 + 3\Delta x \\ \vdots &\qquad \vdots \\ x_{i-1} &= x_0 + (i-1)\Delta x \qquad (1) \\ x_i &= x_0 + i\Delta x \qquad (2) \\ \vdots &\qquad \vdots \\ x_n &= x_0 + n\Delta x \end{aligned}$$

The set of points $P = \{x_0, x_1, x_2, \cdots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, dividing the interval $[a, b]$ into n sub-intervals of the form $[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$ is called a *partition* of $[a, b]$. The partition of $[a, b]$ becomes a *regular partition* of $[a, b]$ if each sub-interval shares the same width.

Author's Interruption. **Spoiler alert:** As we move through the text, we will be required to utter "the ith sub-interval" over and over again. Therefore, it is always a good idea to acquaint ourselves with the ith sub-interval, that is,

$$[x_{i-1}, x_i]$$

from the beginning, where

$$\begin{aligned} x_{i-1} = x_0 + (i-1)\Delta x \quad &\Longrightarrow \quad \text{the left-endpoint of the } i\text{th sub-interval} \\ x_i = x_0 + i\Delta x \quad &\Longrightarrow \quad \text{the right-endpoint of the } i\text{th sub-interval} \end{aligned}$$

Make sure to take a minute or two to comprehend that.

Now that we have formed the sub-intervals for an arbitrary function $f(x)$, we now use them to form rectangles that let us approximate the area under the curve. This can result in either an overestimation or an underestimation of the exact area, depending on how the rectangles are constructed.

We will first consider an underestimation. As discussed earlier, underestimation occurs when the rectangles are constructed so that they lie entirely beneath the

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curve. For that, as we saw before, the left-endpoint of each sub-interval is used to determine the height of the rectangles. Therefore, we start our discussion with the left-endpoint approximation and following that, we will discuss the right-endpoint approximation.

But before that, it is important to note the ith sub-interval, that is,

$$[x_{i-1}, x_i] = [x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$$

for $i = 1, 2, 3, \cdots, n$, respectively. With that in mind, we start discussing the left-endpoint approximation, as follows.

$\cdots$ / Subsec. 5.1.3 / Segment 5.1.3.1 : The Left-Endpoint Approximation

Since $[x_{i-1}, x_i] = [x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$ for $i = 1, 2, 3, \cdots, n$, respectively. Therefore, we construct a rectangle on the ith interval $[x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$, which have a width of Δx and a height equal to the function value at the left endpoint of $[x_{i-1}, x_i]$, that is, $f(x_{i-1})$. Therefore, the area yields,

$$\text{Area} = \text{height} \times \text{width} = f(x_{i-1}) \cdot \Delta x$$

Now, by summing all these areas for $i = 1, 2, 3, \cdots, n$, we get an underestimation of the exact area A. We denote this underestimation as L_n (where L represents that we're using the left endpoint approximation and the subscript n represents the number of rectangles we're using). Thus, for $i = 1, 2, 3, \cdots, n$, the underestimation L_n yields,

$$\begin{aligned}
L_n &= \underbrace{\big(f(x_0) \cdot \Delta x\big)}_{1^{\text{st}}\text{ rectangle}} + \underbrace{\big(f(x_1) \cdot \Delta x\big)}_{2^{\text{nd}}\text{ rectangle}} + \cdots + \underbrace{\big(f(x_{n-2}) \cdot \Delta x\big)}_{(n-1)^{\text{th}}\text{ rectangle}} + \underbrace{\big(f(x_{n-1}) \cdot \Delta x\big)}_{n^{\text{th}}\text{ rectangle}} \\
&= \Delta x \cdot \Big(f(x_0) + f(x_1) + \cdots + f(x_{n-2}) + f(x_{n-1})\Big) \\
&= \Delta x \cdot \sum_{i=1}^{n} f\left(x_{i-1}\right) \qquad \text{for } i = 1, 2, 3, \cdots, n \\
&= \sum_{i=1}^{n} f\left(x_{i-1}\right) \cdot \Delta x
\end{aligned}$$

Therefore, for any arbitrary function $f(x)$, we have the following definition,

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Definition 5.1.1. (The Left-Endpoint Approximation)
Let $f(x)$ be a continuous, non-negative function defined over an interval $[a, b]$. Then, the left-endpoint approximation L_n to underestimate the exact area A bounded under the curve of $f(x)$ and over the x-axis, from $x = a$ to $x = b$, is defined as,

$$L_n = \sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x$$

where, n is the number of rectangles, x_{i-1} is the left-endpoint of the ith sub-interval, and Δx is the width of each sub-interval.

Similar to the left-endpoint, we discuss the right-endpoint approximation below.

$\cdots$ / Subsec. 5.1.3 / Segment 5.1.3.2 : The Right-Endpoint Approximation

Since $[x_{i-1}, x_i] = [x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$ for $i = 1, 2, 3, \cdots, n$, respectively. Therefore, we construct a rectangle on the ith interval $[x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$, which have a width of Δx and a height equal to the function value at the right endpoint of $[x_{i-1}, x_i]$, that is, $f(x_i)$. Therefore, the area yields,

$$\text{Area} = \text{height} \times \text{width} = f(x_i) \cdot \Delta x$$

Now, by summing all these areas for $i = 1, 2, 3, \cdots, n$, we get an overestimation of the exact area A. We denote this overestimation as R_n (where R represents that we're using the right endpoint approximation and the subscript n represents the number of rectangles we're using). Thus, for $i = 1, 2, 3, \cdots, n$, the overestimation R_n yields,

$$\begin{aligned}
R_n &= \underbrace{\left(f(x_1) \cdot \Delta x\right)}_{1^{\text{st}}\text{ rectangle}} + \underbrace{\left(f(x_2) \cdot \Delta x\right)}_{2^{\text{nd}}\text{ rectangle}} + \cdots + \underbrace{\left(f(x_{n-1}) \cdot \Delta x\right)}_{(n-1)^{\text{th}}\text{ rectangle}} + \underbrace{\left(f(x_n) \cdot \Delta x\right)}_{n^{\text{th}}\text{ rectangle}} \\
&= \Delta x \cdot \Big(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\Big) \\
&= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right) \qquad \text{for } i = 1, 2, 3, \cdots, n \\
&= \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x
\end{aligned}$$

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Therefore, for any arbitrary function $f(x)$, we have the following definition,

Definition 5.1.2. (The Right-Endpoint Approximation)
Let $f(x)$ be a continuous, non-negative function defined over an interval $[a, b]$. Then, the right-endpoint approximation R_n to overestimate the exact area A bounded under the curve of $f(x)$ and over the x-axis, from $x = a$ to $x = b$, is defined as,

$$R_n = \sum_{i=1}^{n} f\left(x_i\right)\Delta x$$

where, n is the number of rectangles, x_i is the right-endpoint of the ith sub-interval, and Δx is the width of each sub-interval.

Having generalized the left and right endpoint approximations, we can now proceed from where we left off.

To quickly recap, recall that we aimed to mathematically show that increasing the number of rectangles under the curve of an arbitrary function $f(x)$ results in a closer approximation of the exact area A. This was possible by considering the exact area A as being bounded between two estimates—one representing underestimation and the other overestimation—and showing that these bounds get closer to the exact area as the number of rectangles increases. However, since the manual calculation of these approximations was tedious, we generalized the process for both underestimation and overestimation. Through this generalization, we found that for any arbitrary function $f(x)$, the underestimation (denoted as L_n) of the bounded area A is given by,

$$L_n = \sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x$$

Similarly, the overestimation (denoted as R_n) of the bounded area A is given by,

$$R_n = \sum_{i=1}^{n} f\left(x_i\right)\Delta x$$

This suggests that for any arbitrary function $f(x)$, the exact bounded area A is as follows,

$$\sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x < A < \sum_{i=1}^{n} f\left(x_i\right)\Delta x$$

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where, n represents the number of rectangles.

Now, if we go back to our previous problem of finding the exact area under the curve of $f(x) = \frac{x^2}{4}$ and above the x-axis, from $x = 1$ to $x = 4$; we see that the exact area A can be determined by the following inequality,

$$\sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x < A < \sum_{i=1}^{n} f\left(x_i\right)\Delta x$$

However, in order to use the inequality above, we have to first determine the value of x_{i-1}, x_i, and Δx.

Since the entire width of the interval $[1, 4]$ is 3, therefore, dividing this interval into n sub-interval gives each sub-interval a width of

$$\Delta x = \frac{3}{n}$$

Now, to determine x_{i-1} and x_i, recall that,

from equation (1),	x_{i-1}	$= 1 + (i-1) \cdot \frac{3}{n}$	$= \frac{3(i-1)+n}{n}$
from equation (2),	x_i	$= 1 + i \cdot \frac{3}{n}$	$= \frac{3i+n}{n}$

With that in hand, we now began to calculate the underestimation $L_n = \sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x$ to the exact area under the curve of the function $f(x) = \frac{x^2}{4}$ as follows,

$$\begin{aligned}
L_n &= \sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x \\
&= \sum_{i=1}^{n} f\left(\frac{3\left(i-1\right)+n}{n}\right) \cdot \Delta x \\
&= \sum_{i=1}^{n} \frac{\left(\frac{3\left(i-1\right)+n}{n}\right)^2}{4} \cdot \frac{3}{n} && \text{since, } f(x) = \frac{x^2}{4} \\
&= \sum_{i=1}^{n} \frac{\frac{\left(3\left(i-1\right)+n\right)^2}{n^2}}{4} \cdot \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{\frac{\left(3\left(i-1\right)\right)^2 + 2 \cdot 3\left(i-1\right) \cdot n + n^2}{n^2}}{4} \cdot \frac{3}{n}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{9(i-1)^2 + 6n\,(i-1) + n^2}{4n^2} \cdot \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{3\left(9(i-1)^2 + 6n\,(i-1) + n^2\right)}{4n^3} \\
&= \frac{3}{4n^3} \cdot \sum_{i=1}^{n} \left(9(i-1)^2 + 6n\,(i-1) + n^2\right) \\
&= \frac{3}{4n^3} \left(\sum_{i=1}^{n} 9(i-1)^2 + \sum_{i=1}^{n} 6n\,(i-1) + \sum_{i=1}^{n} n^2\right) \\
&= \frac{3}{4n^3} \left(9 \cdot \sum_{i=1}^{n} (i-1)^2 + 6n \cdot \sum_{i=1}^{n} (i-1) + \sum_{i=1}^{n} n^2\right) \\
&= \frac{3}{4n^3} \left(9 \cdot \sum_{i=1}^{n} \left(i^2 - 2i + 1\right) + 6n \cdot \sum_{i=1}^{n} (i-1) + \sum_{i=1}^{n} n^2\right) \\
&= \frac{3}{4n^3} \left(9 \left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 1\right) + 6n \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) + \sum_{i=1}^{n} n^2\right) \\
&= \frac{3}{4n^3} \left(9 \left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) + 6n \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) + \sum_{i=1}^{n} n^2\right) \\
&= \frac{3}{4n^3} \left(9 \left(\frac{n\,(n+1)\,(2n+1)}{6} - 2\frac{n\,(n+1)}{2} + n\right) + 6n \left(\frac{n\,(n+1)}{2} - n\right) + n^3\right)
\end{aligned}
$$

Now, simplifying $\frac{3}{4n^3} \left(9 \left(\frac{n\,(n+1)\,(2n+1)}{6} - 2\frac{n\,(n+1)}{2} + n\right) + 6n \left(\frac{n\,(n+1)}{2} - n\right) + n^3\right)$ yields,

$$\boxed{L_n = \frac{3(14n^2 - 15n + 3)}{8n^2}}$$

Similarly, we calculate the overestimation $R_n = \sum_{i=1}^{n} f\,(x_i)\Delta x$ to the exact area A under the curve of the function $f(x) = \frac{x^2}{4}$ as follows,

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n} f\,(x_i)\Delta x \\
&= \sum_{i=1}^{n} f\left(\frac{3i+n}{n}\right) \cdot \Delta x \\
&= \sum_{i=1}^{n} \frac{\left(\frac{3i+n}{n}\right)^2}{4} \cdot \frac{3}{n} \qquad \text{since, } f(x) = \frac{x^2}{4}
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{\frac{(3i+n)^2}{n^2}}{4} \cdot \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{\frac{(3i)^2 + 2 \cdot 3i \cdot n + n^2}{n^2}}{4} \cdot \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{9i^2 + 6ni + n^2}{4n^2} \cdot \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{3\left(9i^2 + 6ni + n^2\right)}{4n^3} \\
&= \frac{3}{4n^3} \cdot \sum_{i=1}^{n} 9i^2 + 6ni + n^2 \\
&= \frac{3}{4n^3} \left(\sum_{i=1}^{n} 9i^2 + \sum_{i=1}^{n} 6ni + \sum_{i=1}^{n} n^2 \right) \\
&= \frac{3}{4n^3} \left(9 \cdot \sum_{i=1}^{n} i^2 + 6n \cdot \sum_{i=1}^{n} i + \sum_{i=1}^{n} n^2 \right) \\
&= \frac{3}{4n^3} \left(9 \cdot \frac{n\left(n+1\right)\left(2n+1\right)}{6} + 6n \cdot \frac{n\left(n+1\right)}{2} + n^3 \right)
\end{aligned}
$$

Now, simplifying $\frac{3}{4n^3} \left(9 \cdot \frac{n\left(n+1\right)\left(2n+1\right)}{6} + 6n \cdot \frac{n\left(n+1\right)}{2} + n^3 \right)$ yields,

$$\boxed{R_n = \frac{3(14n^2 + 15n + 3)}{8n^2}}$$

This suggests that we have derived two formulas for underestimation L_n and overestimation R_n to the exact area A, where n represents the number of rectangles. That is, the exact area A bounded under the curve of $f(x) = \frac{x^2}{4}$ and above the x-axis, from $x = 1$ to $x = 4$, is given by the following inequality,

$$\frac{3(14n^2 - 15n + 3)}{8n^2} < A < \frac{3(14n^2 + 15n + 3)}{8n^2}$$

To verify these formulas, set $n = 1$ and $n = 3$ to see if the resulting inequalities match with the previously manually calculated inequalities: $0.75 < A < 12$ for $n = 1$ and $3.5 < A < 7.25$ for $n = 3$.

If $n = 1$, then

$$\frac{3\left(14n^2 - 15n + 3\right)}{8n^2} < A < \frac{3\left(14n^2 + 15n + 3\right)}{8n^2}$$

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$$\frac{3\left(14\cdot 1^2-15\cdot 1+3\right)}{8\cdot 1^2}<A<\frac{3\left(14\cdot 1^2+15\cdot 1+3\right)}{8\cdot 1^2}$$
$$0.75<A<12$$

Which matches with the previously manually done inequality. Similarly, if $n=3$, then,

$$\frac{3\left(14n^2-15n+3\right)}{8n^2}<A<\frac{3\left(14n^2+15n+3\right)}{8n^2}$$
$$\frac{3\left(14\cdot 3^2-15\cdot 3+3\right)}{8\cdot 3^2}<A<\frac{3\left(14\cdot 3^2+15\cdot 3+3\right)}{8\cdot 3^2}$$
$$3.5<A<7.25$$

Which also matches with the previously manually done inequality of the exact area A.

Notice that as the number of rectangles n increases, both the underestimation and overestimation seem to converge to a particular value around 5. That is,

For $n=1$ rectangles:

1. Overestimation: $\dfrac{3(14n^2+15n+3)}{8n^2}=\dfrac{3(14\cdot 1^2+15\cdot 1+3)}{8\cdot 1^2}=12$
2. Underestimation: $\dfrac{3(14n^2-15n+3)}{8n^2}=\dfrac{3(14\cdot 1^2-15\cdot 1+3)}{8\cdot 1^2}=0.75$
3. Inequality: $0.75<A<12$

For $n=3$ rectangles:

1. Overestimation: $\dfrac{3(14n^2+15n+3)}{8n^2}=\dfrac{3(14\cdot 3^2+15\cdot 3+3)}{8\cdot 3^2}=7.25$
2. Underestimation: $\dfrac{3(14n^2-15n+3)}{8n^2}=\dfrac{3(14\cdot 3^2-15\cdot 3+3)}{8\cdot 3^2}=3.5$
3. Inequality: $3.5<A<7.25$

For $n=9$ rectangles:

1. Overestimation: $\dfrac{3(14n^2+15n+3)}{8n^2}=\dfrac{3(14\cdot 9^2+15\cdot 9+3)}{8\cdot 9^2}=5.89$
2. Underestimation: $\dfrac{3(14n^2-15n+3)}{8n^2}=\dfrac{3(14\cdot 9^2-15\cdot 9+3)}{8\cdot 9^2}=4.6389$
3. Inequality: $4.639<A<5.889$

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For $n = 81$ rectangles:

1. Overestimation: $\dfrac{3(14n^2 + 15n + 3)}{8n^2} = \dfrac{3(14 \cdot 81^2 + 15 \cdot 81 + 3)}{8 \cdot 81^2} = 5.3196$
2. Underestimation: $\dfrac{3(14n^2 - 15n + 3)}{8n^2} = \dfrac{3(14 \cdot 81^2 - 15 \cdot 81 + 3)}{8 \cdot 81^2} = 5.1807$
3. Inequality: $5.1807 < A < 5.3196$

For $n = 6561$ rectangles:

1. Overestimation: $\dfrac{3(14n^2 + 15n + 3)}{8n^2} = \dfrac{3(14 \cdot 6561^2 + 15 \cdot 6561 + 3)}{8 \cdot 6561^2} = 5.2509$
2. Underestimation: $\dfrac{3(14n^2 - 15n + 3)}{8n^2} = \dfrac{3(14 \cdot 6561^2 - 15 \cdot 6561 + 3)}{8 \cdot 6561^2} = 5.249$
3. Inequality: $5.249 < A < 5.2509$

In fact, as $n \to \infty$, both the underestimation and overestimation converge to the exact area A. This can be proven by taking the limit of the underestimation and overestimation as $n \to \infty$.

For that, let us first take the limit of the underestimation as $n \to \infty$ as follows.

$$\begin{aligned}
\lim_{n\to\infty} \frac{3(14n^2 - 15n + 3)}{8n^2} &= \lim_{n\to\infty} \frac{3}{8} \cdot \frac{14n^2 - 15n + 3}{n^2} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{14n^2 - 15n + 3}{n^2} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{\dfrac{14n^2}{n^2} - \dfrac{15n}{n^2} + \dfrac{3}{n^2}}{\dfrac{n^2}{n^2}} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{14 - \dfrac{15}{n} + \dfrac{3}{n^2}}{1} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \left(14 - \frac{15}{n} + \frac{3}{n^2}\right) \\
&= \frac{3}{8} \cdot (14 - 0 + 0) \\
&= \frac{42}{8} \\
&= 5.25
\end{aligned}$$

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Similarly, taking the limit of the overestimation as $n \to \infty$ yields,

$$\begin{aligned}
\lim_{n\to\infty} \frac{3(14n^2+15n+3)}{8n^2} &= \lim_{n\to\infty} \frac{3}{8} \cdot \frac{14n^2+15n+3}{n^2} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{14n^2+15n+3}{n^2} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{\frac{14n^2}{n^2} + \frac{15n}{n^2} + \frac{3}{n^2}}{\frac{n^2}{n^2}} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \frac{14 + \frac{15}{n} + \frac{3}{n^2}}{1} \\
&= \frac{3}{8} \cdot \lim_{n\to\infty} \left(14 + \frac{15}{n} + \frac{3}{n^2}\right) \\
&= \frac{3}{8} \cdot (14+0+0) \\
&= \frac{42}{8} \\
&= 5.25
\end{aligned}$$

Now, since both the underestimation and overestimation approach a value of 5.25 as the number of rectangles n increases indefinitely, that is,

$$\lim_{n\to\infty} \frac{3(14n^2-15n+3)}{8n^2} = \lim_{n\to\infty} \frac{3(14n^2+15n+3)}{8n^2} = 5.25 = A$$

therefore the exact area A bounded under the curve of $f(x) = \frac{x^2}{4}$ and above the x-axis, from $x = 1$ to $x = 4$ is 5.25.

Having determined the exact area A in the previously mentioned problem, it's important to notice something significant. In finding the exact area, we constructed an infinite number of rectangles under the curve and observed that both the underestimation and overestimation approach the same value. This implies that to determine the exact area, the use of just one approximation is sufficient. Moreover, the approximation doesn't need to be specifically an overestimation or an underestimation—it can be any approximation when the number of rectangles is unbounded.

Previously, to perform an underestimation, we used the left endpoint of the ith sub-interval, and for an overestimation, we used the right endpoint. This was necessary because using endpoints other than the right or left would neither completely result in an underestimation nor an overestimation. However,

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we later realized that as the number of rectangles n increases indefinitely, both approximations converge to the same value. Therefore, it is now unnecessary to use only the right or left endpoints as long as the number of rectangles n increases indefinitely.

It is the best moment to introduce the Riemann Sum and define the area under a curve in terms of that in the following subsection.

Chap. 5 / Sec. 5.1 / Subsec. 5.1.4 : Riemann Sum and the Definition of the Area Under a Curve

The Riemann sum resolves a few restrictions. For example, we can now choose any point $x_i^* \in [x_{i-1}, x_i]$ to determine the height of the rectangle over the ith sub-interval, for $i = 1, 2, 3, \cdots, n$, as shown below,

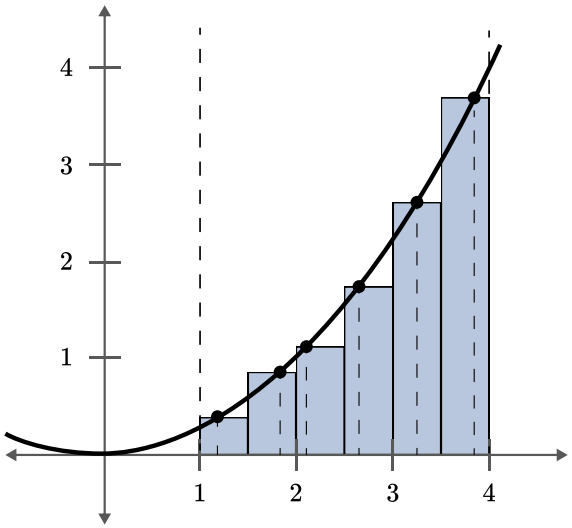

Figure 5.11

Doing so would not change the limit of the approximation, if the number of rectangle $n \to \infty$ is considered. Therefore, let $f(x_i^*)$ represent the height of the rectangle over the ith sub-interval $[x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$, where, $x_i^* \in [x_{i-1}, x_i]$ is any arbitrary point. Then, the area of the rectangle over that sub-interval yields,

$$\text{height} \times \text{width} = f(x_i^*) \cdot \Delta x \qquad \text{for } i = 1, 2, 3, \cdots, n$$

Now, summing these quantities for $i = 1, 2, 3, \cdots, n$, we get an approximation

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to the exact area A, which is called the Riemann Sum:

$$\begin{aligned}
A &\approx \underbrace{\left(f(x_1^*)\cdot\Delta x\right)}_{1^{\text{st}}\text{ rectangle}} + \underbrace{\left(f(x_2^*)\cdot\Delta x\right)}_{2^{\text{nd}}\text{ rectangle}} + \cdots + \underbrace{\left(f(x_{n-1}^*)\cdot\Delta x\right)}_{(n-1)^{\text{th}}\text{ rectangle}} + \underbrace{\left(f(x_n^*)\cdot\Delta x\right)}_{n^{\text{th}}\text{ rectangle}} \\
&= \Delta x \cdot \underbrace{\left(f(x_1^*) + f(x_2^*) + \cdots + f(x_{n-1}^*) + f(x_n^*)\right)}_{n\text{ term summations}} \\
&= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i^*\right) \qquad \text{for } i = 1, 2, 3, \cdots, n \\
&= \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x \\
\therefore A &\approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x
\end{aligned}$$

The approximation $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$ is called the Riemann sum, named after German mathematician Georg Friedrich Bernhard Riemann. Therefore, we have the following definition of the approximation above.

Definition 5.1.3. (Riemann Sum)
Let $f(x)$ be a continuous, non-negative function defined over an interval $[a, b]$. Then, the Riemann sum S is defined as

$$S = \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

where, n is the number of rectangles, x_i^* is any point in the ith sub-interval $[x_{i-1}, x_i]$, and Δx is the width of each sub-interval.

The point $x_i^* \in [x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$, used in the approximation above is called a "sample point". Now, based on the definition of the Riemann sum, we define the area under a curve. That is, if the number of rectangles $n \to \infty$, then the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

represents the exact area A in terms of the Riemann sum. As a result, we can define the exact area A bounded under the curve of $f(x)$ and above the x-axis, from $x = a$ to $x = b$, in terms of the Riemann sum as follows.

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Definition 5.1.4. (Area Under a Curve)
Let $f(x)$ be a continuous, non-negative function defined over an interval $[a, b]$, and let $S = \sum_{i=1}^{n} f(x_i^*)\Delta x$ be the Riemann sum for $f(x)$. Then, the exact area A under the curve of $f(x)$ and over the x-axis, from $x = a$ to $x = b$, is defined as,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Now, notice that the current definition of the "area under a curve" comes with a great limitation: it only applies when the function $f(x)$ is non-negative. In other words, for this definition to apply, the curve has to lie entirely above the x-axis. If any part of the curve lies below the x-axis, the definition above can't be applied there.

To resolve the issue, the concept of "Definite Integral" comes into play. Definite integral expands the definition of the area under a curve to include regions bounded by curves that can lie below the x-axis by considering both positive and negative regions bounded by the curve of $f(x)$. To make this distinction, we introduce the idea of "net signed area," where the curve lying above the x-axis contributes positively, and below the x-axis contributes negatively, as discussed in the following section.

———————The End of Section 5.1
The Area Under a Curve

§ 5.2 The Definite Integral

In the preceding section we defined the area under a curve in terms of Riemann sums as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

However, that definition required the function $f(x)$ to be non-negative. Since not all functions satisfy these conditions, we may need to remove these requirements. As a result, $f(x)$ can now sometimes be negative. But this raises the question: how do we interpret Riemann sums when $f(x)$ is negative at some or all of its

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points over the interval $[a, b]$? The answer is given through the concept of net signed area, as discussed below.

Chap. 5 / Sec. 5.2 / Subsec. 5.2.1 : Net Signed Area and the Definition of the Definite Integral

To understand the net signed area, approximate the area bounded by the curve of $f(x) = -x^2$ from $x = 0$ to $x = 2$, using $n = 4$ rectangles.

Since the interval is $[0, 2]$, therefore after dividing the interval into n sub-intervals results in the width of each sub-interval being $\Delta x = \dfrac{2-0}{4} = 0.5$. Therefore, the sub-intervals yields,

$$[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$$

Now, to use the Riemann sum approximation, we choose the sample points (which can be any points within each respective sub-interval),

$$\begin{aligned} x_1^* &= 0.3 \in [0, 0.5] \\ x_2^* &= 0.7 \in [0.5, 1] \\ x_3^* &= 1.2 \in [1, 1.5] \\ x_4^* &= 1.9 \in [1.5, 2] \end{aligned}$$

As a result, the resulting Riemann sums yield the following,

$$\begin{aligned} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{4} f\left(x_i^*\right) \cdot (0.5) \\ &= f\left(0.3\right) \cdot (0.5) + f\left(0.7\right) \cdot (0.5) + f\left(1.2\right) \cdot (0.5) + f\left(1.9\right) \cdot (0.5) \\ &= \left(f\left(0.3\right) + f\left(0.7\right) + f\left(1.2\right) + f\left(1.9\right)\right) \cdot (0.5) \\ &= (-0.09 - 0.49 - 1.44 - 3.61) \cdot (0.5) \\ &= (-5.63) \cdot (0.5) \\ &= -2.815 \end{aligned}$$

Here, all the values of the height $f(x_i^*)$ are negative. As a result, the Riemann sum—which approximates the area under the curve—turns out to be also negative. However, if the Riemann sum is used to approximate the "area" under the curve, then how can the area—being a physical measurement—be a negative

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quantity?

The key point to realize here is that the Riemann sum, in this case, does not approximate the area in the physical sense. Instead, it approximates the "net signed area," that is, a *difference of areas* that accounts for both the positive and the negative contributions of areas.

Although the positive contribution of area implies the actual positive area, the negative contribution of area does not literally imply a "negative area," as if there were an absence or void of space between the curve and the x-axis. Instead, it represents the *negative* of the actual area bounded above the curve and below the x-axis (see the red shaded region in Figure 5.12). This happens because, in the Riemann sum, the height of the ith rectangle is determined by the the values of the function $f(x_i^*)$, where $x_i^* \in [x_{i-1}, x_i]$. Now, if $f(x_i^*)$ falls below the x-axis, then the height of the ith rectangle, that is, $-f(x_i^*)$ represents the *negative* of the actual height $|f(x_i^*)|$, and thus the product, $-f\,(x_i^*)\Delta x$, represents the *negative* of the actual product $|f\,(x_i^*)\Delta x|$, which is the area of the ith rectangle when $f(x)$ falls below the x-axis.

Now, times occur when $f(x)$ is positive over some parts of $[a, b]$ and negative over other parts of $[a, b]$. In such cases, the Riemann sum approximates the positive area when $f(x)$ is positive and negative of the area when $f(x)$ is negative. Therefore, the Riemann sum over the interval $[a, b]$ approximates the sum of the positive areas that lie above the x-axis and the negative of the areas that lie beneath the x-axis, which is precisely what we refer to as the net signed area, as shown below.

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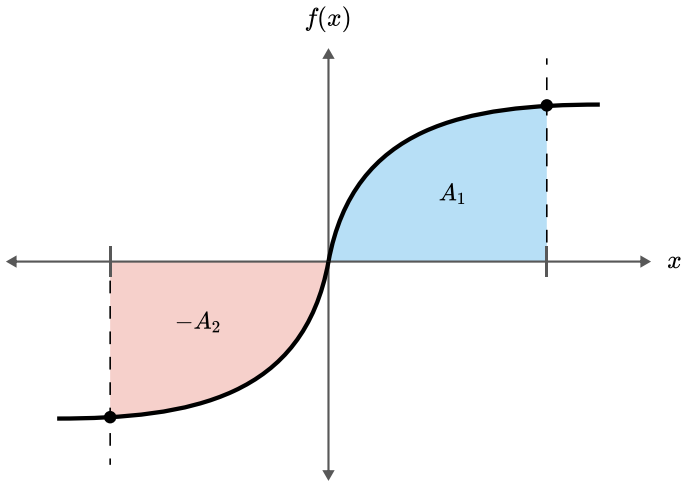

Figure 5.12

In the figure above, the area bounded under the curve of $f(x)$ and above the x-axis is A_1. Similarly, the area bounded above the curve of $f(x)$ and below the x-axis is $-A_2$. Now, the Riemann sum, as the number of rectangles $n \to \infty$, represents the sum of these areas A_1 and $-A_2$. Therefore, we write,

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= A_1 + (-A_2) \\ &= A_1 - A_2\end{aligned}$$

The quantity $A_1 - A_2$ is what we call the net signed area, which can be positive, negative, or even zero.

If there's multiple positive and negative of the areas; for example, suppose, $A_1, A_2, A_3, \cdots, A_n$ are the areas bounded under the curve of $f(x)$ and above the x-axis, and $-B_1, -B_2, -B_3, \cdots, -B_k$ are the areas bounded above the curve of $f(x)$ and below the x-axis. Then the net signed area yields,

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \\ &= (A_1 + A_2 + A_3 + \cdots + A_n) + ((-B_1) + (-B_2) + (-B_3) + \cdots + (-B_k)) \\ &= (A_1 + A_2 + A_3 + \cdots + A_n) + (-B_1 - B_2 - B_3 - \cdots - B_k) \\ &= (A_1 + A_2 + A_3 + \cdots + A_n) + (-(B_1 + B_2 + B_3 + \cdots + B_k)) \\ &= (A_1 + A_2 + A_3 + \cdots + A_n) - (B_1 + B_2 + B_3 + \cdots + B_k) \\ &= \text{sum of areas above the } x\text{-axis} - \text{sum of areas below the } x\text{-axis}\end{aligned}$$

As a result, we have the following definition of the net signed area.

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Definition 5.2.1. (Net Signed Area)
Let R be the region bounded by the graph of $f(x)$ and the x-axis over the interval $[a, b]$. Then the net signed area of R is defined as,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = R_1 - R_2$$

where, R_1 is the sum of areas of R that lie above the x-axis and R_2 is the sum of areas of R that lie below the x-axis.

This suggests that the Riemann sum approximates the net signed area bounded by the curve of $f(x)$ and the x-axis over the interval $[a, b]$. This approximation gets closer to the exact net signed area as the number of rectangles $n \to \infty$.

Notice that as the number of rectangles $n \to \infty$, the width of each rectangle $\Delta x \to 0$. This idea is intuitive but important to emphasize because the Riemann sum we have used so far considers regular partitions where all sub-intervals have the same width Δx. However, in reality, the interval $[a, b]$ can be divided into sub-intervals of unequal widths. In such cases, each sub-interval has a different width, denoted as Δx_i for $i = 1, 2, 3, \ldots, n$.

When $n \to \infty$ in the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, all $\Delta x_i \to 0$. We denote the largest width among the sub-intervals as $\max \Delta x_i$. This implies that if $\max \Delta x_i \to 0$, then all Δx_i also approach 0, for $i = 1, 2, 3, \cdots, n$. Therefore, we write the net signed area bounded by the curve of $f(x)$ and the x-axis over a given interval in terms of the following Riemann sum,

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

where, $\max \Delta x_i$ is the largest width among all sub-intervals. This idea works because as $\max \Delta x_i \to 0$, all $\Delta x_i \to 0$, for $x_i = 1, 2, 3, \cdots, n$. As all $\Delta x_i \to 0$, n must also increase indefinitely.

With that in mind, it is finally the time that we define the "definite integral" to expand the previously discussed definition of the area under a curve.

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Definition 5.2.2. (The Definite Integral)
Let $f(x)$ be a function defined on an interval $[a, b]$. If the Riemann sum,

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$$

exists, then the definite integral of $f(x)$, from $x = a$ to $x = b$, symbolically,

$$\int_a^b f\left(x\right) dx$$

is defined as the following limit:

$$\int_a^b f\left(x\right) dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$$

and $f(x)$ is said to be integrable on $[a, b]$.

The definition above suggests that for a definite integral to exist, the limit used to define that integral must first exist. The precise interpretation of this limit is that,

For all $\varepsilon > 0$, there exists an integer N such that,

$$\left| \int_a^b f\left(x\right) dx - \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i \right| < \varepsilon$$

for all $n > N$ and for all $x_i^* \in [x_{i-1}, x_i]$.

The notation for the definite integral has historical origins and is read as "the definite integral from a to b of $f(x)$ dee x." The symbol $\int$ is called the integral sign and was introduced by Gottfried Wilhelm Leibniz. This symbol represents an elongated S, as in the word 'sum' and it was selected because an integral represents the limit of a sum. The numbers a and b attached to the integral sign are called the lower and upper limits of integration, respectively. The function $f(x)$ is called the integrand, and dx indicates that the integration is performed with respect to x. The figure below illustrates these components.

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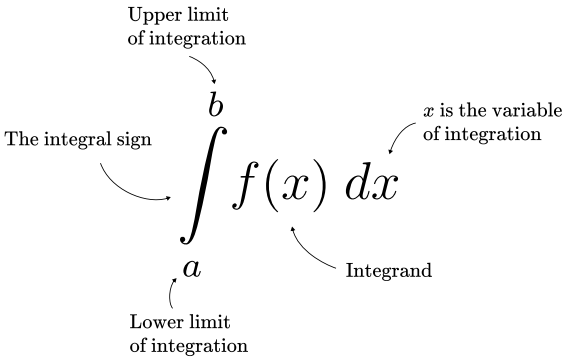

The variable of integration is a dummy variable, it has no effect on the computation of the integral, as long as it does not conflict with other variables that are in use. That is, the following notation all means the same thing.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \int_a^b f(s)\,ds$$

It is important to note that we have defined the definite integral for integrable functions but not all functions fall into this criteria. A function $f(x)$ is integrable if $f(x)$ is continuous on the interval $[a, b]$. Additionally, if $f(x)$ is bounded on $[a, b]$, meaning that there exists a number M such that $|f(x)| < M$ for all $x \in [a, b]$ and it has only a finite number of discontinuities, then $f(x)$ is also integrable on $[a, b]$. This implies that even if a function is not continuous, it can still be integrable on $[a, b]$. As a result, we have the following theorem.

Theorem 5.2.1. (Integrability of Continuous Functions)
If $f(x)$ is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then $f(x)$ is integrable on $[a, b]$.

The theorem above suggests that if $f(x)$ is continuous on $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ represents the net signed area bounded by the curve $f(x)$ and the x-axis over that given interval. This means that the integral takes into account both the areas above the x-axis (which are considered positive) and those below the x-axis (which are considered negative). As a result, the definite integral $\int_a^b f(x)\,dx$ gives a single numerical value, which corresponds to the total quantity accumulated, considering the signs of the areas.

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A few examples to evaluate the definite integral using the definition above are provided below.

► **Example 5.2.1.** Evaluate $\int_0^1 2xdx$

Solution. In order to evaluate the definite integral, we divide the interval $[0, 1]$ into n sub-intervals of equal width. The width of each sub-interval yields,

$$\frac{1-0}{n} = \frac{1}{n}$$

Now, we let the ith sample point x_i^* be the right endpoint of each sub-interval for calculation simplicity. Then, we get,

$$\begin{aligned} x_i^* &= x_0 + i\Delta x \\ &= 0 + i \cdot \frac{1}{n} \\ \therefore x_i^* &= \frac{i}{n} \end{aligned}$$

Therefore, the Riemann sum for the function $f(x) = 2x$ yields,

$$\begin{aligned} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i &= \sum_{i=1}^{n} 2\left(\frac{i}{n}\right) \cdot \frac{1}{n} \\ &= \frac{2}{n^2} \cdot \sum_{i=1}^{n} i \\ &= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{n} \end{aligned}$$

Now, as the number of rectangles $n \to \infty$, then,

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} \\ &= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} \\ &= 1 \end{aligned}$$

Therefore, $\int_0^1 2xdx = 1$ ∎

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► **Example 5.2.2.** Evaluate $\int_0^2 x^2 dx$

Solution. In order to evaluate the definite integral, we divide the interval $[0, 2]$ into n sub-intervals of equal width. The width of each sub-interval yields,

$$\frac{2-0}{n} = \frac{2}{n}$$

Now, we let the ith sample point x_i^* be the right endpoint of each sub-interval for calculation simplicity. Then, we get,

$$\begin{aligned} x_i^* &= x_0 + i\Delta x \\ &= 0 + i \cdot \frac{2}{n} \\ &= \frac{2i}{n} \end{aligned}$$

Therefore, the Riemann sum for the function $f(x) = x^2$ yields,

$$\begin{aligned} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} \\ &= \sum_{i=1}^{n} \frac{4i^2}{n^2} \cdot \frac{2}{n} \\ &= \sum_{i=1}^{n} \frac{8i^2}{n^3} \\ &= \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{4(n+1)(2n+1)}{3n^2} \end{aligned}$$

Now, as the number of rectangles $n \to \infty$, then,

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i &= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} \\ &= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} \\ &= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} \end{aligned}$$

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$$
\begin{aligned}
&= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2}} \\
&= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{1} \\
&= \frac{4}{3} \cdot \left(\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{3}{n} + \lim_{n\to\infty} \frac{1}{n^2} \right) \\
&= \frac{8}{3}
\end{aligned}
$$

Therefore, $\int_0^2 x^2 dx = \frac{8}{3}$. ∎

Having discussed the definite integral and its evaluation, we now discuss its properties in the next subsection.

Chap. 5 / Sec. 5.2 / Subsec. 5.2.2 : The Properties of the Definite Integral

The properties of the definite integral acts as a set of rules that govern how definite integrals can be manipulated and evaluated. We start our discussion by introducing the the reversal of limits of integration and the identical limits of integration, as follows.

· · · / Subsec. 5.2.2 / Segment 5.2.2.1 : The Reversal of Limits of Integration and Identical Limits of Integration

Recall that we have defined the definite integral as:

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where we indirectly assumed that the lower limit of integration a is less than the upper limit of integration b, that is, $a < b$. However, it is certainly possible that $a > b$. In such cases, the width of each sub-interval changes from Δx_i to $-\Delta x_i$, for $i = 1, 2, 3, \cdots, n$, in the Riemann sum on the right side of the equation. This is due to the fact that $a - b < 0$ is divided into n sub-intervals, rather than

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$b - a > 0$. As a result, we have,

$$\begin{aligned}\int_b^a f(x)\,dx &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,(-\Delta x_i) \\ &= -\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \\ &= -\int_a^b f(x)\,dx \\ \therefore \int_b^a f(x)\,dx &= -\int_a^b f(x)\,dx\end{aligned}$$

Therefore, we have the following theorem.

Theorem 5.2.2. (The Reversal of Limits of Integration)
Let $f(x)$ be an integrable function defined over an interval $[a, b]$, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Now, if the lower limit of integration a is equal to the upper limit b (i.e., $a = b$), then $\Delta x = 0$. This happens because the interval $a - a = 0$ is divided into n sub-intervals, which is simply $\dfrac{0}{n} = 0$. Therefore,

$$\begin{aligned}\int_a^a f(x)\,dx &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \cdot 0 \\ &= 0 \\ \therefore \int_a^a f(x)\,dx &= 0\end{aligned}$$

Therefore, we have the following theorem.

Theorem 5.2.3. (Identical Limits of Integration)
Let $f(x)$ be an integrable function defined over an interval $[a, b]$, then

$$\int_a^a f(x)\,dx = 0$$

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· · · / Subsec. 5.2.2 / Segment 5.2.2.2 : Integrals Involving a Sum or Difference of Functions

If $f(x)$ and $g(x)$ are integrable functions, then the definite integral of the function $f(x) + g(x)$ is given by the following theorem.

Theorem 5.2.4. (Integrals Involving a Sum of Functions)
Let $f(x)$ and $g(x)$ be integrable functions defined over an interval $[a, b]$, then

$$\int_a^b \Big(f\left(x\right) + g\left(x\right)\Big)dx = \int_a^b f\left(x\right)dx + \int_a^b g\left(x\right)dx$$

Proof. We will proof this directly.

$$\begin{aligned}
\int_a^b \left(f(x) + g(x)\right) dx &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \left(f\left(x_i^*\right) + g\left(x_i^*\right)\right) \Delta x_i \\
&= \lim_{\max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i + \sum_{i=1}^{n} g\left(x_i^*\right) \Delta x_i \right) \\
&= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i + \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} g\left(x_i^*\right) \Delta x_i \\
\therefore \int_a^b \left(f(x) + g(x)\right) dx &= \int_a^b f(x)dx + \int_a^b g(x)dx
\end{aligned}$$

This completes the proof. ∎

Similar to the theorem above, we also have the following theorem for integrals involving a difference of functions.

Theorem 5.2.5. (Integrals Involving a Difference of Functions)
Let $f(x)$ and $g(x)$ be integrable functions defined over an interval $[a, b]$, then

$$\int_a^b \Big(f\left(x\right) - g\left(x\right)\Big)dx = \int_a^b f\left(x\right)dx - \int_a^b g\left(x\right)dx$$

Proof. We will proof this directly.

$$\int_a^b \left(f(x) - g(x)\right) dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \left(f\left(x_i^*\right) - g\left(x_i^*\right)\right) \Delta x_i$$

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$$= \lim_{\max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i - \sum_{i=1}^{n} g\left(x_i^*\right) \Delta x_i \right)$$

$$= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i - \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} g\left(x_i^*\right) \Delta x_i$$

$$\therefore \int_a^b \left(f(x) - g(x)\right) dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

This completes the proof. ∎

⋯ / Subsec. 5.2.2 / Segment 5.2.2.3 : Integrals Involving a Constant

If $f(x)$ is an integrable function defined over an interval $[a, b]$, then the definite integral of the function $cf(x)$, where c is a constant, is given by the following theorem.

Theorem 5.2.6. (Integrals Involving a Constant)
Let $f(x)$ be an integrable function defined over an interval $[a, b]$, then

$$\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$$

Proof. We will proof this directly.

$$\int_a^b cf(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} cf(x)\Delta x_i$$

$$= \lim_{\max \Delta x_i \to 0} \left(c \cdot \sum_{i=1}^{n} f(x)\Delta x_i \right)$$

$$= \lim_{\max \Delta x_i \to 0} c \cdot \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x)\Delta x_i$$

$$= c \cdot \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x)\Delta x_i$$

$$= c \cdot \int_a^b f(x)dx$$

This completes the proof. ∎

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· · · / Subsec. 5.2.2 / Segment 5.2.2.4 : Integrals over Multiple Sub-intervals

If $f(x)$ is an integrable function defined over an interval $[a, b]$ and there exists a point c in between a and b, then the integral on $[a, b]$ may be split into the following two integrals over $[a, c]$ and $[c, b]$, respectively.

Theorem 5.2.7. (Integrals over Multiple Sub-intervals)
Let $f(x)$ and $g(x)$ be integrable functions defined over an interval $[a, b]$ and c be a constant such that $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The theorem above is pretty clear when seen visually as follows.

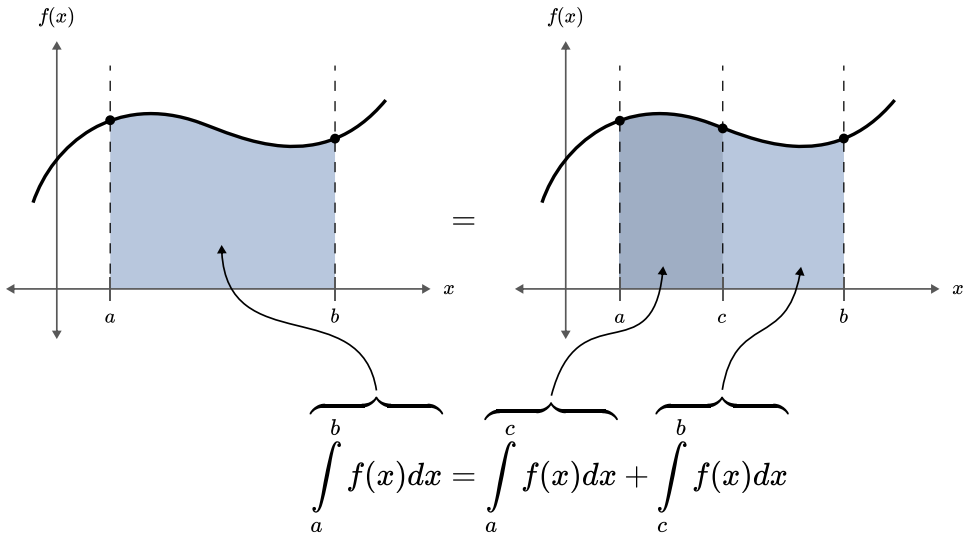

Figure 5.13

· · · / Subsec. 5.2.2 / Segment 5.2.2.5 : Integrals Involving Absolute Values

In addition to the properties mentioned above, it is important to consider the property of the definite integral involving an absolute value. For example, consider the definite integral of the form $\int_a^b |f(x)|\,dx$. The absolute value function, $|f(x)|$, represents the distance of the function $f(x)$ from the x-axis, ensuring that the result is always non-negative. This means that $|f(x)|$ returns the magnitude

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of $f(x)$ without regard to whether $f(x)$ is positive or negative.

As a result, the integral $\int_a^b |f(x)|\,,dx$ calculates the "total area" between the curve $f(x)$ and the x-axis over the interval $[a, b]$. This includes the area above the x-axis where $f(x)$ is positive, as well as the area below the x-axis where $f(x)$ is negative, since the absolute value converts negative values of $f(x)$ to positive. Therefore, this integral provides the sum of the areas of both regions, treating both as positive contributions.

That is, the definite integral $\int_a^b f(x)dx$ represents the following net signed area.

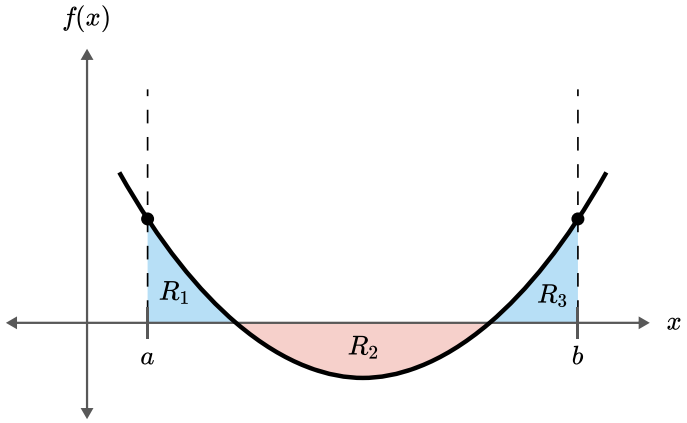

Figure 5.14

While the definite integral $\int_a^b |f(x)|\,dx$ represents the following area under the curve of $|f(x)|$.

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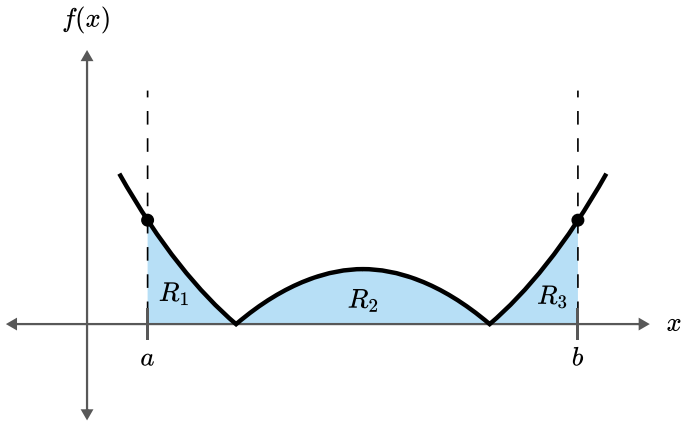

Figure 5.15

Therefore, we have the following theorem.

Theorem 5.2.8. (Integrals Involving Absolute Values)
Let $f(x)$ be a function such that $|f(x)|$ is integrable on the interval $[a, b]$, then the definite integral,

$$\int_a^b |f(x)|\, dx$$

represents the sum of areas of the region bounded the function $f(x)$ and the x-axis, from $x = a$ to $x = b$.

Having discussed the properties above, they are now summarized below for a quick reference.

Property 5.2.1. Let $f(x)$ and $g(x)$ be integrable functions on an interval $[a, b]$ and k be a constant, then

1. $\displaystyle\int_a^a f(x)\, dx = 0$
2. $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
3. $\displaystyle\int_a^b k \cdot f(x)\, dx = k \cdot \int_a^b f(x)\, dx$

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4. $\displaystyle\int_a^b \Big(f(x)+g(x)\Big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

5. $\displaystyle\int_a^b \Big(f(x)-g(x)\Big)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

6. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

A few examples of the properties above are provided below.

► **Example 5.2.3.** If $\displaystyle\int_2^1 x^2dx = -\frac{7}{3}$, then what is $\displaystyle\int_1^2 x^2dx$?

Solution. If $\displaystyle\int_2^1 x^2dx = -\frac{7}{3}$, then by the property above,

$$\begin{aligned}\int_1^2 x^2dx &= -\int_2^1 x^2dx\\ &= -\left(-\frac{7}{3}\right)\\ &= \frac{7}{3}\end{aligned}$$

Therefore, $\displaystyle\int_1^2 x^2dx = \frac{7}{3}$ ∎

► **Example 5.2.4.** If $\displaystyle\int_0^1 f(x)dx = e$ and $\displaystyle\int_0^1 g(x)dx = \pi$, then what is $\displaystyle\int_0^1 (\pi f(x) + eg(x))\,dx$

Solution. If $\displaystyle\int_0^1 f(x)dx = e$ and $\displaystyle\int_0^1 g(x)dx = \pi$, then by the properties above,

$$\begin{aligned}\int_0^1 (\pi f(x) + eg(x))\,dx &= \pi\int_0^1 f(x)dx + e\int_0^1 g(x)dx\\ &= \pi e + e\pi\end{aligned}$$

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$$= 2e\pi$$

Therefore, $\int_0^1 (\pi f(x) + eg(x))\,dx = 2e\pi.$ ∎

———————The End of Section 5.2
The Definite Integral

§ 5.3 Numerical Integration

In the previous sections, we used rectangles to approximate the area of a region bounded by a curve and the x-axis. In those approximations, each rectangle generally had an equal width, but their heights varied depending on the type of approximation used. In the right endpoint approximation, the height of each rectangle was determined by the function's value at the right endpoint of each sub-interval. Similarly, in the left endpoint approximation, the height was determined by the function's value at the left endpoint of each sub-interval.

We then introduced the Riemann sum, where the height of each rectangle could be based on any point within each sub-interval. This method is valid based on the fact that all these approximations converge to the same limit as the number of rectangles approaches infinity.

Following these methods, we now introduce a few new approximations.

Chap. 5 / Sec. 5.3 / Subsec. 5.3.1 : The Midpoint Rule

In the Midpoint Rule, we approximate the area bounded by the curve and the x-axis using rectangles as done before. However, this time, each rectangle's height is determined by the value of the function at the midpoint m_i of each sub-interval. For instance, for the ith sub-interval $[x_{i-1}, x_i]$, the height of the rectangle is the value of the function at the midpoint $m_i = \dfrac{x_i + x_{i-1}}{2}$, as shown below.

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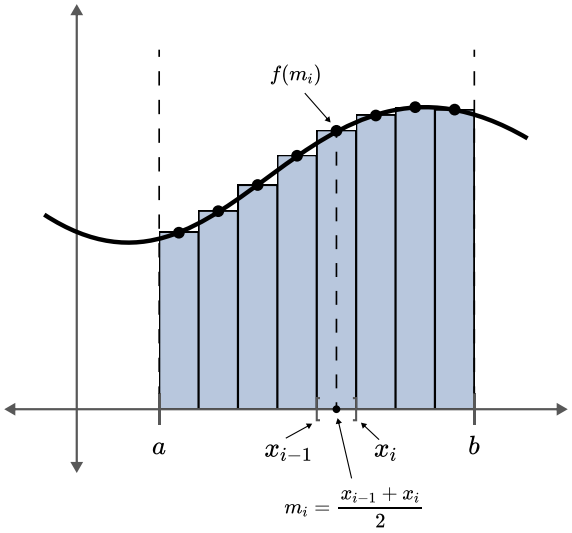

Figure 5.16

That being said, we now generalize this approximation. Suppose we want to approximate the area bounded by the curve of $f(x)$ and the x-axis from $x = a$ to $x = b$, by the Midpoint rule.

In order to do that we partition the interval $[a, b]$ into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$, yielding the following sub-intervals:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$$

This implies that the ith sub-interval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \cdots, n$. Recall that the midpoint rule approximates the area by rectangles whose height is determined by the function's value at the midpoint of each sub-interval. We denote such height as $f(m_i)$, where m_i represents the midpoint of the ith sub-interval, that is, $\dfrac{x_{i-1} + x_i}{2}$. Therefore, the area of each rectangle yields the following:

$$\text{height} \times \text{width} = f(m_i)\Delta x = f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

Now, summing these quantities for $i = 1, 2, 3, \cdots, n$, we get the following sum denoted by M_n.

$$\begin{aligned} M_n &= \underbrace{f\left(m_1\right)\Delta x}_{1^{\text{st}}\text{ rectangle}} + \underbrace{f\left(m_2\right)\Delta x}_{2^{\text{nd}}\text{ rectangle}} + \cdots + \underbrace{f\left(m_n\right)\Delta x}_{n^{\text{th}}\text{ rectangle}} \\ &= \Big(f\left(m_1\right) + f\left(m_2\right) + \cdots + f\left(m_n\right)\Big)\Delta x \end{aligned}$$

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$$= \Delta x \cdot \sum_{i=1}^{n} f(m_i) \qquad \text{for } i = 1, 2, \cdots, n$$
$$= \sum_{i=1}^{n} f(m_i)\, \Delta x$$
$$= \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x$$

The Midpoint rule approximation gets better and better as the number of rectangles n increases. This forms the following definition.

Definition 5.3.1. (Midpoint Rule)
Let M_n represent the Midpoint rule approximation to the definite integral $\int_a^b f(x)\, dx$ on the interval $[a, b]$. Then,

$$M_n = \sum_{i=1}^{n} f(m_i)\, \Delta x$$

where, $m_i = \dfrac{x_{i-1} + x_i}{2}$ is the midpoint of the ith interval, $[x_{i-1}, x_i]$, and Δx is the width of each sub-interval.

A few examples to demonstrate the definition above are provided below.

► **Example 5.3.1.** Use the Midpoint Rule with 6 sub-intervals to estimate $\int_0^{\pi} \sin(x) dx$

Solution. In order to approximate, we divide the interval $[0, \pi]$ into $n = 6$ sub0intervals of equal width. That is,

$$\frac{\pi - 0}{6} = \frac{\pi}{6}$$

Thus, we have the following sub-intervals,

$$\left[0, \frac{\pi}{6}\right], \left[\frac{\pi}{6}, \frac{\pi}{3}\right], \left[\frac{\pi}{3}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{2\pi}{3}\right], \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right], \left[\frac{5\pi}{6}, \pi\right]$$

Thus, we have the following midpoints of each sub-intervals,

$$m_1 = \frac{0 + \frac{\pi}{6}}{2} = \frac{\pi}{12}$$

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$$m_2 = \frac{\frac{\pi}{6} + \frac{\pi}{3}}{2} = \frac{\pi}{4}$$
$$m_3 = \frac{\frac{\pi}{3} + \frac{\pi}{2}}{2} = \frac{5\pi}{12}$$
$$m_4 = \frac{\frac{\pi}{2} + \frac{2\pi}{3}}{2} = \frac{7\pi}{12}$$
$$m_5 = \frac{\frac{2\pi}{3} + \frac{5\pi}{6}}{2} = \frac{3\pi}{4}$$
$$m_6 = \frac{\frac{5\pi}{6} + \pi}{2} = \frac{11\pi}{12}$$

Now, we approximate the integral as follows,

$$\begin{aligned}
\int_0^\pi \sin(x)dx &\approx \sum_{i=1}^{6} f(m_i)\Delta x \\
&= f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x + f(m_5)\Delta x + f(m_6)\Delta x \\
&= \Delta x \left[f\left(\frac{\pi}{12}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{5\pi}{12}\right) + f\left(\frac{7\pi}{12}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{11\pi}{12}\right) \right] \\
&= \frac{\pi}{6} \cdot \left[\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{11\pi}{12}\right) \right] \\
&\approx 2.02303
\end{aligned}$$

Therefore, $\int_0^\pi \sin(x)dx \approx 2.02303$. ∎

Notice that in all of our approximations discussed so far, we used rectangles to estimate areas under curves. However, it is certainly not the case that we are limited solely to the rectangles to approximate the area—other geometric shapes can also be used instead. A common alternative of using rectangles is to the trapezoids. We discuss this in the following subsection.

Chap. 5 / Sec. 5.3 / Subsec. 5.3.2 : The Trapezoidal Rule

In the Trapezoidal rule approximation, we approximate the net area bounded by the curve of $f(x)$ and the x-axis on an interval $[a, b]$ with trapezoids instead of rectangles. For that, recall that the area of a trapezoid with a height of h and the lengths of the two parallel sides a and b, that is,

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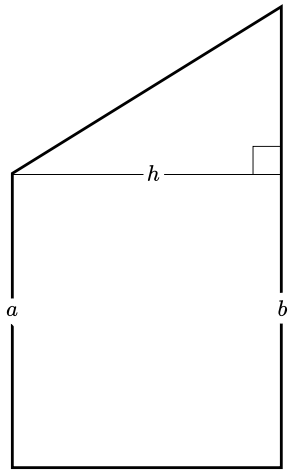

Figure 5.17

is given by the following formula,

$$\text{area} = \left(\frac{a+b}{2}\right) h$$

Now, in order to use the Trapezoidal rule approximation to approximate the net area bounded by the curve of $f(x)$ and the x-axis on the interval $[a, b]$, we divide the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = \dfrac{b-a}{n}$. This forms the following sub-intervals:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$$

This implies that the ith sub-intervals is $[x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$. We now construct trapezoids on the ith sub-interval for $i = 1, 2, 3, \cdots, n$, as shown below.

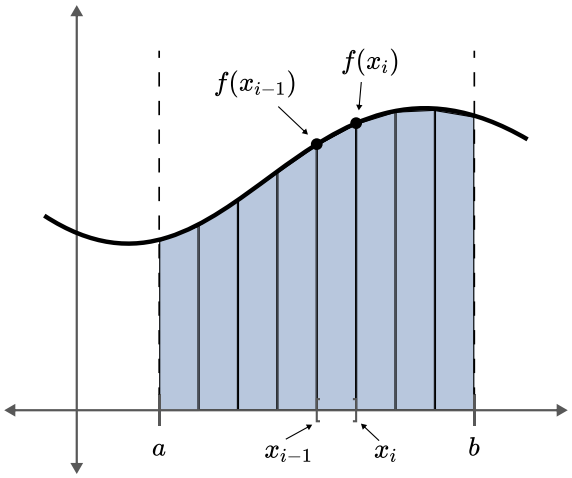

Figure 5.18

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The height of each trapezoid is equal to the width of each sub-interval, that is, $h = \Delta x = \dfrac{b-a}{n}$ and the lengths of each trapezoid's sides a and b are $f(x_{i-1})$ and $f(x_i)$, respectively, for $i = 1, 2, 3, \cdots$. Therefore, the area of the ith trapezoid yields,

$$\text{area} = \left(\frac{f\left(x_{i-1}\right) + f\left(x_i\right)}{2}\right)\Delta x$$

Now, summing these quantities for $i = 1, 2, 3, \cdots, n$, we get the following sum denoted by T_n.

$$\begin{aligned}
T_n &= \underbrace{\left(\frac{f\left(x_0\right) + f\left(x_1\right)}{2}\right)\Delta x}_{1^{\text{st}}\text{ trapezoid}} + \underbrace{\left(\frac{f\left(x_1\right) + f\left(x_2\right)}{2}\right)\Delta x}_{2^{\text{nd}}\text{ trapezoid}} + \cdots + \underbrace{\left(\frac{f\left(x_{n-1}\right) + f\left(x_n\right)}{2}\right)\Delta x}_{n^{\text{th}}\text{ trapezoid}} \\
&= \left(\left(\frac{f\left(x_0\right) + f\left(x_1\right)}{2}\right) + \left(\frac{f\left(x_1\right) + f\left(x_2\right)}{2}\right) + \cdots + \left(\frac{f\left(x_{n-1}\right) + f\left(x_n\right)}{2}\right)\right)\Delta x \\
&= \left(\left(\frac{f\left(x_0\right)}{2} + \frac{f\left(x_1\right)}{2}\right) + \left(\frac{f\left(x_1\right)}{2} + \frac{f\left(x_2\right)}{2}\right) + \cdots + \left(\frac{f\left(x_{n-1}\right)}{2} + \frac{f\left(x_n\right)}{2}\right)\right)\Delta x \\
&= \left(\frac{f\left(x_0\right)}{2} + \underbrace{\frac{f\left(x_1\right)}{2} + \frac{f\left(x_1\right)}{2}}_{f(x_1)} + \cdots + \underbrace{\frac{f\left(x_{n-1}\right)}{2} + \frac{f\left(x_{n-1}\right)}{2}}_{f(x_{n-1})} + \frac{f\left(x_n\right)}{2}\right)\Delta x \\
&= \left(\frac{f\left(x_0\right)}{2} + \underbrace{f\left(x_1\right) + f\left(x_2\right) + \cdots + f\left(x_{n-1}\right)}_{\sum_{i=1}^{n-1} f(x_i)} + \frac{f\left(x_n\right)}{2}\right)\Delta x \\
&= \left(\frac{f\left(x_0\right)}{2} + \sum_{i=1}^{n-1} f\left(x_i\right) + \frac{f\left(x_n\right)}{2}\right)\Delta x
\end{aligned}$$

The Trapezoidal rule approximation gets better and better as the number of rectangles n increases. This leads to the following definition.

Definition 5.3.2. (Trapezoidal Rule) Let $f(x)$ be continuous over an interval $[a, b]$. The Trapezoidal Rule to approximate $\displaystyle\int_a^b f\left(x\right)dx$ is given by,

$$T_n = \frac{\Delta x}{2}\Big(f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + \cdots + 2f\left(x_{n-1}\right) + f\left(x_n\right)\Big)$$

Where, $\displaystyle\lim_{n\to\infty} T_n = \int_a^b f\left(x\right)dx$

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A few examples of the definition above are presented below.

▶ **Example 5.3.2.** Use the trapezoidal rule with 4 sub-intervals to estimate $\int_0^1 x^2 dx$.

Solution. The width of each sub-interval is,

$$\Delta x = \frac{1-0}{4} = 0.25$$

This implies that,

$$\begin{aligned} x_0 &= 0 + 0 \cdot 0.25 = 0 \\ x_1 &= 0 + 1 \cdot 0.25 = 0.25 \\ x_2 &= 0 + 2 \cdot 0.25 = 0.5 \\ x_3 &= 0 + 3 \cdot 0.25 = 0.75 \\ x_4 &= 0 + 4 \cdot 0.25 = 1 \end{aligned}$$

Now, compute the function values,

$$\begin{aligned} f(0) &= 0^2 = 0 \\ f(0.25) &= (0.25)^2 = 0.0625 \\ f(0.5) &= (0.5)^2 = 0.25 \\ f(0.75) &= (0.75)^2 = 0.5625 \\ f(1) &= (1)^2 = 1 \end{aligned}$$

Using the composite trapezoidal rule,

$$\begin{aligned} \int_0^1 x^2 dx &\approx \frac{0.25}{2}\left(f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)\right) \\ &= \frac{0.25}{2}\left(0 + 2(0.0625 + 0.25 + 0.5625) + 1\right) \\ &= \frac{0.25}{2}\left(0 + 2(0.875) + 1\right) \\ &= \frac{0.25}{2}\left(2.75\right) \\ &= 0.34375 \end{aligned}$$

Therefore, $\int_0^1 x^2 dx \approx 0.34375$ ∎

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THE END OF SECTION 5.3
Numerical Integration

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Chapter 6

The Connection Between Differential and Integral Calculus

As of now, we have been introduced to two main branches of calculus: differential calculus and integral calculus. As discussed in the previous chapters, differential calculus concerns itself with the rate of change of a quantity as in the slopes of curves, while integral calculus focuses on the accumulation of quantities as in the areas under curves.

These two concepts may seem unrelated—how could the slope of a curve be possibly connected to the area under it? However, the relation between these counter-intuitive ideas can ultimately be shown by the Fundamental Theorem of Calculus, which we aim to study shortly. But, before we dive into that, though, it is essential to understand the concepts of antiderivatives and indefinite integrals. Therefore, we begin our discussion with these foundational concepts.

§ 6.1 Antidifferentiation

The concept of an inverse operation is a key concept in mathematics. For example, in arithmetic, addition and subtraction are inverse of one another as one operation can reverse the effect of the other. Similarly, multiplication and division are inverses of one another as applying one operation can be undone by applying the other.

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Similarly, in calculus, since differentiation is an operation, and like other operations, it has an inverse. This inverse of differentiation is known as *antidifferentiation*. While differentiation is the process of finding the derivative of a function, antidifferentiation is the process of finding an *antiderivative* of a function. This prompts us to study the concept of antiderivative.

We discuss the concept of antiderivative in the following subsection.

Chap. 6 / Sec. 6.1 / Subsec. 6.1.1 : Antiderivative

The concept of an antiderivative of a function is unique in that it is a differentiable function whose derivative is equal to the originally given function. That is, an antiderivative of a given function $f(x)$ is a differentiable function $F(x)$ such that when $F(x)$ is differentiated, it equals the original function $f(x)$.

This concept may seem complicated and, quite frankly, somewhat confusing. Nevertheless, to properly understand the concept, consider the following.

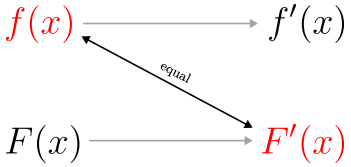
Consider a function $f(x)$. If $f(x)$ is differentiable then its derivative is denoted as $f'(x)$. That is,

$$f(x) \xrightarrow{\text{derivative}} f'(x)$$

Now, suppose another function $F(x)$. If $F(x)$ is differentiable, then its derivative is denoted as $F'(x)$. That is,

$$F(x) \xrightarrow{\text{derivative}} F'(x)$$

Now, here is the connection: $F(x)$ is said to be an *antiderivative* of $f(x)$ if there's this specific relationship between $F'(x)$ and $f(x)$ such that $F'(x) = f(x)$. That is, if



is true, then the following holds,

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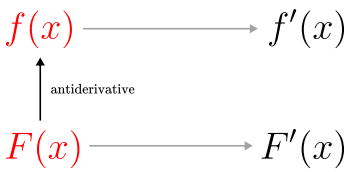

That is, $F(x)$ is an antiderivative of $f(x)$. Based on that, we have the following definition of an antiderivative of a function.

Definition 6.1.1. (Antiderivative)
A function $F(x)$ is an *antiderivative* of $f(x)$ over an interval I, if

$$F'(x) = f(x)$$

for all $x \in I$

The definition above may be better understood by the following examples.

► **Example 6.1.1.** What is an antiderivative of the function $f(x) = 2x$?

Solution. If $F(x) = x^2$, then the derivative is $F'(x) = 2x$. Now, if $f(x) = 2x$, then an antiderivative of $f(x)$ is $F(x) = x^2$, for the reason that $F'(x) = f(x)$. ∎

► **Example 6.1.2.** What is an antiderivative of the function $f(x) = x^2$?

Solution. If $F(x) = \dfrac{x^3}{3}$, then the derivative is

$$\begin{aligned} F'(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}\left(\frac{x^3}{3}\right) \\ &= \frac{1}{3}\cdot 3x^2 \\ &= x^2 \end{aligned}$$

Now, if $f(x) = x^2$, then an antiderivative of $f(x)$ is $F(x) = \dfrac{x^3}{3}$, for the reason that $F'(x) = f(x)$. ∎

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Now, notice that although $\frac{x^3}{3}$ is *an* antiderivative of x^2, it is not—however—*the* antiderivative of x^2. This is because the function $F(x) = \frac{x^3}{3}$ is not the *only* function whose derivative is x^2. Actually, every function of the form $F(x) = \frac{x^3}{3} + C$, where $C \in \mathbb{R}$ a constant, has a derivative of x^2. This is because,

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}\left(\frac{x^3}{3} + C\right)\\ &= \frac{d}{dx}\left(\frac{x^3}{3}\right) + \frac{d}{dx}(C)^{0}\\ &= x^2 + 0\\ &= x^2\end{aligned}$$

This implies that the most *general* antiderivative of x^2 is $\frac{x^3}{3} + C$, where $C \in \mathbb{R}$, and thus we have the following theorem.

Theorem 6.1.1. (The Family of Antiderivatives)
If $F(x)$ is an antiderivative of $f(x)$ over an interval I, then *all* the antiderivative of $f(x)$ over I is of the form,

$$F(x) + C$$

where, $C \in \mathbb{R}$ is an arbitrary constant.

This implies that all the antiderivatives of a function differ only by a constant C and therefore, they belong to one general family of functions, that is, functions of the form $F(x) + C$.

It's fascinating how each differentiation formula, when reversed, transforms into an antidifferentiation formula. For example, as we've just seen, an antiderivative of x^2 is $\frac{x^3}{3}$, suggesting an antiderivative of x^n to be $\frac{x^{n+1}}{n+1}$, where $n \neq -1$, because

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) &= \frac{1}{n+1}\cdot\frac{d}{dx}x^{n+1}\\ &= \frac{1}{n+1}\cdot\left((n+1)\cdot x^{n+1-1}\right)\end{aligned}$$

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$$= \frac{1}{\cancel{n+1}} \cdot \cancel{(n+1)} \cdot x^n$$
$$= x^n \qquad \text{where } n \neq -1$$

Similarly, an antiderivative of $\sin(x)$ is $-\cos(x)$, because $\frac{d}{dx}(-\cos(x)) = -(-\sin(x)) = \sin(x)$. An antiderivative of e^x is e^x, because $\frac{d}{dx}(e^x) = e^x$, and so on. In the table below, a list of functions along with their particular antiderivatives are presented for a quick reference, where each of the particular antiderivatives can be verified by differentiating and matching with the original function.

Table 6.1

Function	Particular antiderivative
0	$C \in \mathbb{R}$
1	x
n	nx
x^n	$\frac{x^{n+1}}{n+1}, n \neq 1$
nx^{n-1}	x^n
$\frac{1}{2\sqrt{x}}$	$\sqrt{x}$
e^x	e^x
$a^x \cdot \ln(a)$	a^x
$\frac{1}{x}$	$\ln(x) = \log_e(x)$
$\frac{1}{x \ln(a)}$	$\log_a(x)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\csc^2(x)$	$-\cot(x)$

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$\tan(x)\sec(x)$	$\sec(x)$
$\cot(x)\csc(x)$	$-\csc(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\cos^{-1}(x)$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$
$-\dfrac{1}{1+x^2}$	$\cot^{-1}(x)$
$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1}(x)$
$-\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\csc^{-1}(x)$

A few examples are provided below.

► **Example 6.1.3.** What is the general antiderivative of $f(x) = \dfrac{1}{2\sqrt{x}}$

Solution. The antiderivative of $f(x) = \dfrac{1}{2\sqrt{x}}$ is,

$$F(x) = \sqrt{x} + C$$

Because,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\sqrt{x} + \frac{d}{dx}C \\ &= \frac{1}{2\sqrt{x}} \\ \therefore F'(x) &= f(x) \end{aligned}$$

Therefore, the antiderivative of $f(x) = \dfrac{1}{2\sqrt{x}}$ is $F(x) = \sqrt{x}+C$, where $C \in \mathbb{R}$. ∎

► **Example 6.1.4.** What is the general antiderivative of $f(x) = \sin(x)$

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Solution. The general antiderivative of $f(x) = \sin(x)$ is,

$$F(x) = -\cos(x) + C$$

Because,

$$\begin{aligned} F'(x) &= \frac{d}{dx}(-\cos(x)) + \frac{d}{dx}C \\ &= \sin(x) \\ \therefore F'(x) &= f(x) \end{aligned}$$

Therefore, the antiderivative of $f(x) = \sin(x)$ is $F(x) = -\cos(x) + C$, where $C \in \mathbb{R}$. ∎

► **Example 6.1.5.** What is the general antiderivative of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

Solution. The general antiderivative of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ is,

$$F(x) = \sin^{-1}(x) + C$$

Because,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\sin^{-1}(x) + \frac{d}{dx}C \\ &= \frac{1}{\sqrt{1-x^2}} \\ \therefore F'(x) &= f(x) \end{aligned}$$

Therefore, the antiderivative of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ is $F(x) = \sin^{-1}(x) + C$, where $C \in \mathbb{R}$. ∎

So far, we have used $F(x)$ to denote a particular antiderivative of the function $f(x)$. However, since there are infinitely many antiderivatives of a single function $f(x)$ that can be written in the form $F(x) + C$, therefore, it becomes necessary to establish a notation that represents all these antiderivatives of a function $f(x)$ collectively. The notation that denotes this concept is referred to as the "Indefinite Integral," we discuss it further below.

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Chap. 6 / Sec. 6.1 / Subsec. 6.1.2 : Indefinite integral

An indefinite integral of a function $f(x)$ represents the family of all antiderivatives, that is, the set of all functions that are an antiderivative of the function $f(x)$.

For historical reasons, the elongated S symbol $\int$—referred to as the *integral sign*—is used to denote the collection of all antiderivatives of a function $f(x)$ in the following manner,

$$\int f(x)dx$$

The notation above is called the *indefinite integral* and read as "the *integral* of $f(x)$ dee x."

Now, observe carefully that the notation for a collection of antiderivatives is the same as the notation for a definite integral $\int_a^b f(x)dx$, except for the core difference that a definite integral is a numerical value that represents the net signed area of the region bounded by the curve of a function $f(x)$ and the x-axis, from $x = a$ to $x = b$. On the other hand, an *indefinite* integral is a family of functions that represents the collection of all antiderivatives of the function $f(x)$. Clearly, a numerical value and a family of functions cannot be the same thing and therefore the definite and indefinite integral belongs to two distinct concepts. This distinction is also reflected in their notation for the limits of integration. That is, a *definite* integral includes limits of integration in its integral sign $\int_a^b$, while an *indefinite* integral does not $\int$.

The reason for using such similar notation for two distinct concepts and causing temporary confusion will become clear once we understand the Fundamental Theorem of Calculus. Without giving too much away, the Fundamental Theorem of Calculus essentially shows that a definite integral can be calculated without using any limit of approximation by the use of an indefinite integral. And, therefore such similar notation for an indefinite integral is strategically chosen for later convenience.

Similar to the notation of a definite integral, the symbol $\int$ is also followed by

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the function $f(x)$ called the *integrad* and the differential dx called the *variable of integration* in the notation of an indefinite integral. The differential dx is simply there to remind the fact that x is the independent variable, similar to the role in the Leibniz notation for derivative, $\frac{d}{dx}f(x)$.

Based on that, we have the following definition of an indefinite integral.

Definition 6.1.2. (Indefinite Integral)
The collection of *all* antiderivatives of a function $f(x)$ is referred to as the indefinite integral of $f(x)$ with respect to x, and is denote as,

$$\int f(x)\,dx$$

where, the symbol $\int$ represents the integral sign. The function $f(x)$ is called the integrand of the integral and x is the variable of integration.

This implies that given a function $f(x)$, the indefinite integral of $f(x)$, that is,

$$\int f\,(x)dx$$

represents all the antiderivative of the function $f(x)$ that are of the form $F(x)+C$. This suggests that

Remark 6.1.1.

$$\int f\,(x)dx = F\,(x) + C$$

Having discussed the concept of antiderivatives and its notation, we now aim to show how Differentiation and Antidifferentiation are indeed inverse operations of each other. For that, recall that for a given function $f(x)$, the family of all antiderivatives of $f(x)$ is represented as,

$$\int f\,(x)dx = F\,(x) + C$$

Since, $f(x) = F'(x)$ by the definition of an antiderivative (as $F(x)$ is an antiderivative of $f(x)$), therefore, we can write,

$$\int F'(x)dx = F\,(x) + C$$

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Now, differentiating both sides of the equation above yields,

$$\begin{aligned}\frac{d}{dx}\int F'(x)dx &= \frac{d}{dx}\left(F\left(x\right)+C\right)\\ &= F'(x)\end{aligned}$$

Since, $f(x) = F'(x)$, therefore, we have the following,

$$\frac{d}{dx}\int f(x)dx = f(x)$$

The result above suggests that when a function $f(x)$ is first antidifferentiated and then differentiated, it results in the same function $f(x)$ or the function $f(x)$ is unchanged.

Similarly, we now show that when a function is first differentiated and then antidifferentiated, it also results in the same function or does not change at all. In order to do that, recall that for a given function $F(x)$, if its antiderivative is $f(x)$ and is differentiated, then we get,

$$\frac{d}{dx}f(x) = f'(x)$$

Now, antidifferentiating both sides of the equation above (that is, finding the family of all antiderivatives of the function $f'(x)$) yields,

$$\int \frac{d}{dx}f(x)dx = \int f'(x)dx$$

Since, $F(x) = f'(x)$ by the definition of an antiderivative (as $f(x)$ is an antiderivative of $F(x)$), therefore,

$$\begin{aligned}\int \frac{d}{dx}f(x)dx &= \int f'(x)dx\\ &= \int F(x)dx\end{aligned} \tag{1}$$

Now, since the notation $\int F(x)dx$ represents the family of all antiderivative of the function $F(x)$ that are of the form $f(x)+C$, therefore, by the definition of an indefinite integral, we write,

$$\int F(x)dx = f(x)+C$$

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And this implies, from equation (1), that

$$\int \frac{d}{dx} f(x)dx = f(x) + C$$

As a result, we have the following theorem,

Theorem 6.1.2.

$$\frac{d}{dx}\int f(x)dx = f(x) \qquad \text{and} \qquad \int \frac{d}{dx} f(x)dx = f(x) + C$$

The theorem above suggests that differentiation and antidifferentiation are indeed inverse operations of each others. Since they are inverses of each other, therefore a few indefinite integrals (i.e., the collection/family of all antiderivatives) can easily be obtained by reversing the derivative of various functions, as listed below.

Note that each indefinite integral listed below can be verified by differentiating the function on the right-hand side of the equation and confirming that it matches the integrand of the corresponding indefinite integral.

Therefore, we have the following informal intuition,

$$\int \text{derivative } dx = \text{antiderivative} + C$$

This intuition suggests that if the "antiderivative" on the right side of the equation is differentiated, it equals the "derivative" in the integrand on the left side.

1. $\displaystyle\int dx = x + C$
2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C$
3. $\displaystyle\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C$
4. $\displaystyle\int a^x dx = \frac{1}{\ln(a)} a^x + C$
5. $\displaystyle\int e^x dx = e^x + C$
6. $\displaystyle\int a^x \ln(a) dx = a^x + C$
7. $\displaystyle\int \frac{dx}{x} = \ln(x) + C$
8. $\displaystyle\int \frac{dx}{x\ln(a)} = \log_a(x) + C$
9. $\displaystyle\int \sin(x)dx = -\cos(x) + C$
10. $\displaystyle\int \cos(x)dx = \sin(x) + C$

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11. $\int \sec^2(x)dx = \tan(x) + C$
12. $\int \csc^2(x)dx = -\cot(x) + C$
13. $\int \sec(x)\tan(x)dx = \sec(x) + C$
14. $\int \csc(x)\cot(x)dx = -\csc(x) + C$
15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
16. $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$
17. $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}(x) + C$

Similar to the constant multiple, addition, and subtraction rules in differentiation, we have the following elementary rules for indefinite integrals (i.e., the collection/family of all antiderivatives).

Theorem 6.1.3. (Elementary Rules for Indefinite Integrals)

1. $\int cf(x)dx = c\int f(x)dx$
2. $\int \Big(f(x) + g(x)\Big)dx = \int f(x)dx + \int g(x)dx$
3. $\int \Big(f(x) - g(x)\Big)dx = \int f(x)dx - \int g(x)dx$

A few examples of the indefinite integral are provided below.

► **Example 6.1.6.** Evaluate $\int 7xdx$

Solution. Using the indefinite integrals, we write,

$$\begin{aligned}\int 7xdx &= 7\int xdx \\ &= \frac{7x^2}{2} + C\end{aligned}$$

Therefore, $\int 7xdx = \frac{7x^2}{2} + C$ ∎

► **Example 6.1.7.** Evaluate $\int \frac{0}{x}dx$

Solution. Using the indefinite integrals, we write,

$$\int \frac{0}{x}dx = \int 0dx$$

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$$\begin{aligned} &= 0 \cdot \int dx \\ &= 0 \cdot x + C \\ &= 0 + C \end{aligned}$$

Therefore, $\int \frac{0}{x} dx = 0 + C$ ∎

► **Example 6.1.8.** Evaluate $\int \left(x^2 + x^3 + x^5\right) dx$

Solution. Using the indefinite integrals, we write,

$$\begin{aligned} \int \left(x^2 + x^3 + x^5\right) dx &= \int x^2 dx + \int x^3 dx + \int x^5 dx \\ &= \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C \end{aligned}$$

Therefore, $\int \left(x^2 + x^3 + x^5\right) dx = \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$ ∎

► **Example 6.1.9.** Evaluate $\int \sin(5x) dx$

Solution. To solve the integral, we apply the Chain rule as follows.

$$\begin{aligned} \frac{d}{dx} \cos(5x) &= -\sin(5x) \cdot \frac{d}{dx}(5x) \\ &= -5 \sin(5x) \end{aligned}$$

Therefore,

$$\begin{aligned} \int -5 \sin(5x) dx &= \int \frac{d}{dx} \cos(5x) dx \\ -5 \cdot \int \sin(5x) dx &= \cos(5x) + C \\ \int \sin(5x) dx &= -\frac{1}{5} \cos(5x) + C \end{aligned}$$

Therefore, $\int \sin(5x) dx = -\frac{1}{5} \cos(5x) + C$ ∎

For the cases illustrated in the example above, let us now generalize the indefinite integral for any constant $a \neq 0$, as follows.

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Remark 6.1.2. (Generalized Indefinite Integrals)

1. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$
2. $\int \frac{dx}{ax} = \frac{1}{a}\ln|x| + C$
3. $\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$
4. $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$
5. $\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C$
6. $\int \csc^2(ax)dx = -\frac{1}{a}\cot(ax) + C$
7. $\int \sec(ax)\tan(ax)dx = \frac{1}{a}\sec(ax) + C$
8. $\int \csc(ax)\cot(ax)dx = -\frac{1}{a}\csc(ax) + C$
9. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
10. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
11. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

► **Example 6.1.10.** Evaluate $\int \frac{dx}{2x}$

Solution. If we let $a = 2$, then the integral matches the form $\int \frac{dx}{ax} = \frac{1}{a}\ln|x| + C$. Therefore,

$$\int \frac{1}{2x}dx = \frac{1}{2}\ln|x| + C$$

Therefore, $\int \frac{1}{2x}dx = \frac{1}{2}\ln|x| + C$. ∎

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► **Example 6.1.11.** Evaluate $\int \frac{3}{\sqrt{25-x^2}}dx$

Solution. If we let $a = 5$, then the integral matches the form $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$. Therefore,

$$\begin{aligned}\int \frac{3}{\sqrt{25-x^2}}dx &= 3 \cdot \int \frac{1}{\sqrt{5^2-x^2}}dx \\ &= 3\sin^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Therefore, $\int \frac{3}{\sqrt{25-x^2}}dx = 3\sin^{-1}\left(\frac{x}{5}\right) + C$ ∎

► **Example 6.1.12.** Evaluate $\int \frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}}dx$

Solution. In order to evaluate the integral $\int \frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}}dx$, we simplify the integrand as follows,

$$\begin{aligned}\frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}} &= \frac{x^2 + x\sqrt{x^2-25}+9}{x\sqrt{x^2-25}\,(9+x^2)} \\ &= \frac{x\sqrt{x^2-25} + \left(x^2+9\right)}{x\sqrt{x^2-25}\,(x^2+9)} \\ &= \frac{\cancel{x\sqrt{x^2-25}}}{\cancel{x\sqrt{x^2-25}}\,(x^2+9)} + \frac{\cancel{x^2+9}}{x\sqrt{x^2-25}\cancel{(x^2+9)}} \\ &= \frac{1}{x^2+9} + \frac{1}{x\sqrt{x^2-25}}\end{aligned}$$

This implies that,

$$\begin{aligned}\int \frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}}dx &= \int \left(\frac{1}{x^2+9} + \frac{1}{x\sqrt{x^2-25}}\right)dx \\ &= \int \frac{1}{x^2+9}dx + \int \frac{1}{x\sqrt{x^2-25}}dx \\ &= \int \frac{1}{x^2+3^2}dx + \int \frac{1}{x\sqrt{x^2-5^2}}dx\end{aligned}$$

The integrals above matches the form $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$ and $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$, for $a = 3$ and $a = 5$, respectively. Therefore, we write,

$$\int \frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}}dx = \int \frac{1}{3^2+x^2}dx + \int \frac{1}{x\sqrt{x^2-5^2}}dx$$

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$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{5}\sec^{-1}\left|\frac{x}{5}\right| + C$$

Therefore, $\displaystyle\int \frac{x^2 + x\sqrt{x^2-25}+9}{9x\sqrt{x^2-25}+x^3\sqrt{x^2-25}}dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{5}\sec^{-1}\left|\frac{x}{5}\right| + C$ ∎

THE END OF SECTION 6.1
Antidifferentiation

§ 6.2 The Fundamental Theorem of Calculus (FTC)

The Fundamental Theorem of Calculus (FTC) is, in a way, the most important of all theorems in calculus. In simple terms, it states that differentiation and integration are complete opposites of each other, much like how addition and subtraction are opposites of one another.

To recap, "differentiation" refers to the process of finding the derivative of a function $f(x)$, which represents the instantaneous rate of change of that function with respect to its independent variable. On the other hand, "integration" is the process of calculating the definite integral of a function $f(x)$, which represents the net signed area between the graph of $f(x)$ and the x-axis over a given interval. This is done by taking the limit of Riemann sum approximations as the number of rectangles n to approach infinity.

Now, up to this point, these two concepts may have seemed unrelated. However, the Fundamental Theorem of Calculus now states that differentiation and integration are actually opposite processes. This might seem contradictory, as we have discussed, in the previous section that, it is antidifferentiation—not integration—that serves as the inverse of differentiation.

However, the key point to realize here is that the Fundamental Theorem of Calculus now allows us to perform integration in terms of antidifferentiation. That is, we can now evaluate definite integrals using antiderivatives, rather than relying on the limit of a Riemann Sum.

It is for this reason that we can say—by the Fundamental Theorem of Calculus—that "integration" and "antidifferentiation" are essentially the same thing when evaluating a definite integral.

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Having introduced the Fundamental Theorem of Calculus and its core idea that integration can be performed through antidifferentiation, we will now explore how this is possible. Specifically, we will derive the first part of the theorem, which then leads to the derivation of the second part. This, in turn, will enable us to evaluate definite integrals using antiderivatives.

In order to do that, consider a function $y = f(t)$ defined over an interval $[a, b]$. If the function $f(t)$ is integrable over $[a, b]$, then the exact area A under the curve $f(t)$ and above the t-axis, over the interval $[a, b]$, that is, from $t = a$ to $t = b$, is given by the definite integral A provided below,

$$A = \int_a^b f\,(t)\,dt$$

Now, say we want to compute the area from $t = a$ to an unspecified bound $t = x$. That is, we want to compute the area A over the interval $[a, x]$, where x is a variable, varying over a set of values greater than a. If that's the case, then the definite integral would now be,

$$\int_a^x f\,(t)\,dt$$

This is shown below,

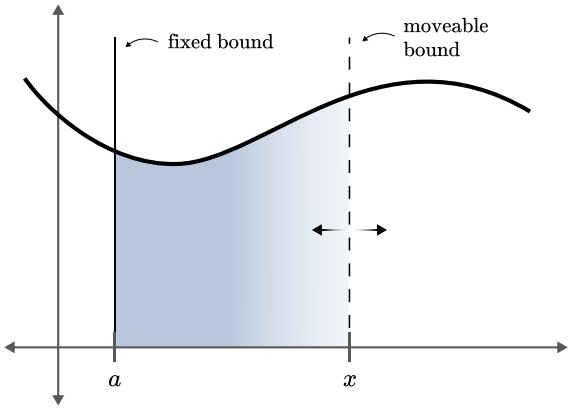

Figure 6.1

Notice here that the definite integral above is not able to produce any value as an area unless the upper limit of integration is specified. That is, since the upper limit of integration is variable x, therefore for each input in the upper limit x, there would be a corresponding output as a numerical value of the definite

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integral. Therefore, the definite integral is now a function of x. We denote this function by $A(x)$. That is,

$$A(x) = \int_a^x f\,(t)\,dt$$

This is the exact area from a to x. Since x is a variable that can take on values greater than a, therefore, an increment $h > 0$ in x would yield,

$$A(x+h) = \int_a^{x+h} f\,(t)\,dt$$

This implies that $A(x+h)$ is the exact area from a to $x+h$ as shown as the shaded region in the figure below,

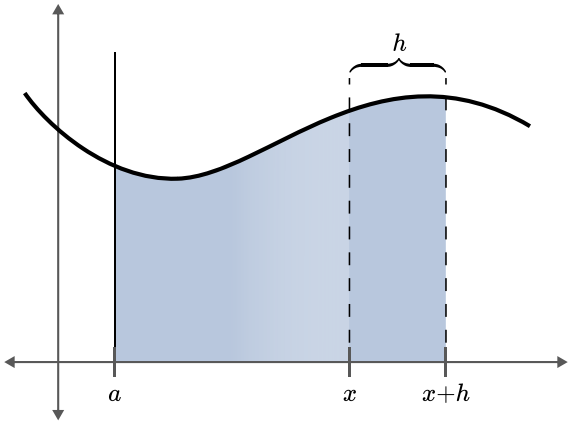

Figure 6.2

Recall that $A(x)$ denotes the exact area from a to x. Therefore, subtracting $A(x+h)$ from $A(x)$ would yield the exact area from x to $x+h$ as shown below,

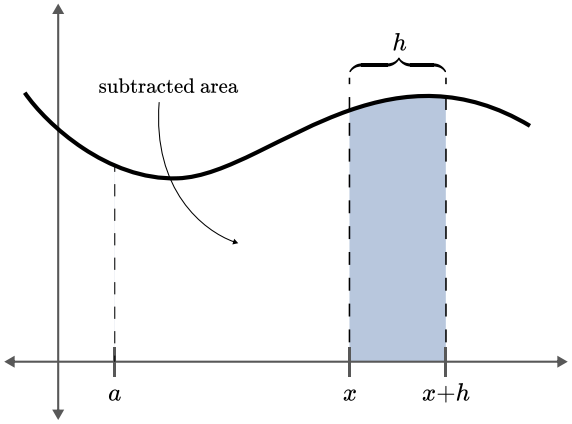

Figure 6.3

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Now, if the increment $h > 0$ is very small then the exact area of $A(x+h) - A(x)$ is,

$$\begin{aligned} A(x+h) - A(x) &\approx \text{area of a thin rectangle at } t = x \\ &= \text{height} \times \text{width} \\ &= f(x) \times h \\ \therefore A(x+h) - A(x) &\approx f(x) \cdot h \end{aligned} \tag{1}$$

as shown below,

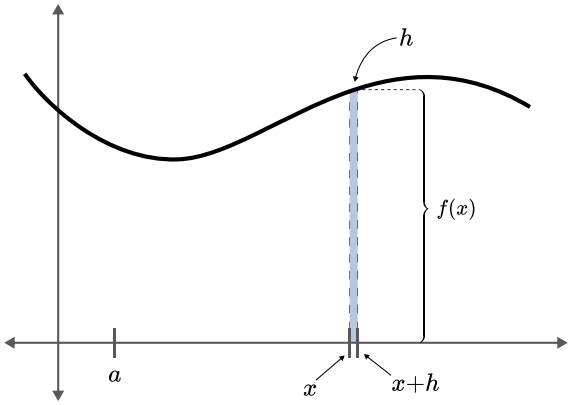

Figure 6.4

Now, dividing both sides by h and then taking the limit as $h \to 0$ yields,

$$\begin{aligned} \frac{A(x+h) - A(x)}{h} &\approx f(x) \\ \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} &= \lim_{h\to 0} f(x) \\ A'(x) &= f(x) \end{aligned} \tag{2}$$

Here, $\lim\limits_{h\to 0} \dfrac{A(x+h) - A(x)}{h} = A'(x)$ by the definition of a derivative. Since $A'(x) = f(x)$ in the equation above, therefore $A(x)$ is an antiderivative of $f(x)$, by the definition of antiderivatives. As a result, we have the following theorem.

Theorem 6.2.1. (The Fundamental Theorem of Calculus, Part I)
Let f be a function continuous on $[a, b]$, then the area function,

$$A(x) = \int_a^x f\,(t)\,dt$$

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where, $a \leq x \leq b$, is continuous on $[a, b]$ and differentiable on (a, b), and

$$A'(x) = f(x)$$

That is, the area function $A(x)$ is an antiderivative of $f(x)$, by the definition of antiderivatives.

The theorem above suggests that, since

$$\begin{aligned} A'(x) &= \frac{d}{dx} A(x) \\ &= \frac{d}{dx} \underbrace{\int_a^x f(t)dt}_{A(x)} \\ &= f(x) \qquad\qquad \text{since } A'(x) = f(x) \\ \therefore A'(x) &= \frac{d}{dx} \int_a^x f(t)dt = f(x) \end{aligned} \tag{3}$$

therefore, the area function $A(x)$ is an antiderivative of $f(x)$ on the interval $[a, b]$. Based on this part of the Fundamental theorem of Calculus, we derive the second part of the theorem by which a definite integral can be evaluated using antiderivatives only.

For that, notice that the area function $A(x)$ above is an antiderivative of $f(x)$. It is not, however, the *only* antiderivative of $f(x)$. From previous discussions, we know that all antiderivatives of a function differ by a constant. So, we let $F(x)$ be another antiderivative of $f(x)$, such that $F(x) = A(x) - C$ or $A(x) = F(x) + C$.

Now, to derive the second part of the theorem, we need to determine the value of C in the equation above. In order to do that, notice that when $x = a$, the area function yields,

$$\begin{aligned} A(x) &= \int_a^a f\left(t\right) dt \\ &= 0 \end{aligned}$$

due to the zero-width property of definite integrals. Therefore,

$$A(a) = 0$$

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$$F(a) + C = 0$$
$$C = -F(a)$$

Now, substituting the value of C in $A(x) = F(x) + C$ yields,

$$A(x) = F(x) + (-F(a))$$
$$= F(x) - F(a)$$

This implies that the definite integral $A(x)$ from $t = a$ to $t = x$ (where x is a variable point) can be calculated by evaluating and subtracting the antiderivative $F(a)$ from $F(x)$, without using the limit of an approximation.

This further implies that to calculate the definite integral $A(x)$ from $t = a$ to $t = x$, that is, over the interval $[a, b]$; we need to evaluate the antiderivative $F(x)$ at both endpoints of the interval $[a, b]$ and subtract $F(a)$ from $F(b)$. Therefore, we have the following theorem.

Theorem 6.2.2. (The Fundamental Theorem of Calculus, Part II)
If $f(x)$ is a continuous function over an interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The part two of the Fundamental Theorem of Calculus above suggests that the definite integral $\int_a^b f(x)\,dx$ can be evaluated in terms of antiderivatives by evaluating $F(x)$ at both endpoints of $[a, b]$, that is, $F(a)$ and $F(b)$, and then subtracting $F(a)$ from $F(b)$.

Notice that if it's shown that the integral in the theorem above is being evaluated, then the following notation is conveniently used.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

That is, the difference $F(b) - F(a)$ is denoted as follows,

$$F(x)\Big]_a^b = F(b) - F(a) \qquad \text{or} \qquad \Big[F(x)\Big]_a^b = F(b) - F(a)$$

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The Fundamental Theorem of Calculus allows us to evaluate definite integrals using antiderivatives, which is based on the concept of derivatives. This creates a similar relationship between integration and antidifferentiation, while creating an inverse relationship between integration and differentiation, as antidifferentiation has always been the inverse operation of differentiation. We discuss this matter further below.

Chap. 6 / Sec. 6.2 / Subsec. 6.2.1 : The Inverse Relationship between Differentiation and Integration

Previously, we have seen, in antidifferentiation, that when a function $f(x)$ is first antidifferentiated and then differentiated, it results in the same function $f(x)$. That is,

$$\frac{d}{dx}\int f(x)dx = f(x)$$

Similarly, we see in the first part of the Fundamental Theorem of Calculus that when a function $f(t)$ is first integrated and then differentiated, it results in the same. That is, from the equation (3), we get,

$$\frac{d}{dx}\int_a^x f(t)dx = f(x)$$

Likewise, we have seen, in antidifferentiation, that when a function is first differentiated and then antidifferentiated, it results in the same function. That is,

$$\int \frac{d}{dx}f(x)dx = f(x) + C$$

where, $f(x)$ is an antiderivative of $\frac{d}{dx}f(x) = f'(x)$ (i.e., simply, $F(x)$).

Similarly, we see in the second part of the Fundamental Theorem of Calculus that when a function $f(x)$ is first differentiated and then integrated over the interval $[a, b]$, it results in the same function $f(x)$, evaluated at both the endpoints a and b. That is,

$$\int_a^b \frac{d}{dx}f(x)dx = f(b) - f(a)$$

where, $f(x)$ is an antiderivative of $\frac{d}{dx}f(x) = f'(x)$ (i.e., simply, $F(x)$)

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This is the reason why the terms "antidifferentiation" and "integration" are often used interchangeably, and both "integration" and "antidifferentiation" are considered the inverse operations of "differentiation" when evaluating a definite integral using the Fundamental Theorem of Calculus.

To demonstrate the Fundamental Theorem of Calculus, a few examples are presented below.

► **Example 6.2.1.** Evaluate $\int_0^1 x^2 dx$

Solution. In order to evaluate the definite integral $\int_0^1 x^2 dx$, we only need to find the antiderivative $F(x)$ of the function $f(x) = x^2$, so that $F'(x)$ yields $f(x)$. We know, from the previous discussion, that an antiderivative $F(x)$ of functions of the form $f(x) = x^n$ is $\frac{x^{n+1}}{n+1}$. Therefore,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Letting $n = 2$ and using the Fundamental Theorem of Calculus yields the following,

$$\begin{aligned}\int_0^1 x^2 dx &= \frac{x^3}{3}\Big]_0^1 \\ &= \frac{(1)^3}{3} - \frac{(0)^3}{3} \\ &= \frac{1}{3}\end{aligned}$$

Therefore, $\int_0^1 x^2 dx = \frac{1}{3}$ ∎

Now, notice carefully that since definite integrals yield a specific numerical value, the constant of integration (C) is not relevant as it cancels out during the evaluation process. That is,

$$\int_a^b f(x)dx = \Big[F(x) + C\Big]_a^b$$

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$$\begin{aligned} &= (F(b) + C) - (F(a) + C) \\ &= F(b) + \not{C} - F(a) - \not{C} \\ &= F(b) - F(a) \end{aligned}$$

Notice also that, since the difference $F(b) - F(a)$ can be written shortly as $F(x)\Big]_a^b$, and since $F(x) = \int f(x)dx$, therefore, we can write,

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \\ &= F(x)\Big]_a^b \\ &= \int f(x)dx\Big]_a^b \end{aligned}$$

This implies that

Remark 6.2.1.

$$\int_a^b f(x)dx = \int f(x)dx\Big]_a^b$$

► **Example 6.2.2.** Evaluate $\int_0^\pi \sin(x)dx$

Solution. We know that,

$$\int \sin(x)dx = -\cos(x) + C$$

Therefore, by the Fundamental Theorem of Calculus, we write the following,

$$\begin{aligned} \int_0^\pi \sin(x)dx &= \Big[-\cos(x)\Big]_0^\pi \\ &= -\left(\cos(\pi) - \cos(0)\right) \\ &= -\left(-1 - 1\right) \\ &= 2 \end{aligned}$$

Therefore, $\int_0^\pi \sin(x)dx = 2$ ∎

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► **Example 6.2.3.** Evaluate $\int_e^{\pi} e^{2x}dx$

Solution. We know that,

$$\int e^{2x}dx = \frac{e^{2x}}{2} + C$$

Therefore, by the Fundamental Theorem of Calculus, we write the following,

$$\begin{aligned}\int_e^{\pi} e^{2x}dx &= \frac{e^{2x}}{2}\Big]_e^{\pi} \\ &= \frac{e^{2\pi}}{2} - \frac{e^{2e}}{2} \\ &= \frac{e^{2\pi} - e^{2e}}{2}\end{aligned}$$

Therefore, $\int_e^{\pi} e^{2x}dx = \frac{e^{2\pi} - e^{2e}}{2}$ ∎

► **Example 6.2.4.** Evaluate $\int_{-e}^{+e} \frac{edx}{x}$

Solution. We know that,

$$\int \frac{dx}{x} = \ln|x| + C$$

Therefore, by the Fundamental Theorem of Calculus, we write the following,

$$\begin{aligned}\int_{-e}^{+e} \frac{edx}{x} &= e\int_{-e}^{+e} \frac{dx}{x} \\ &= e \cdot [\ln|x|]_{-e}^{+e} \\ &= e \cdot (\ln|e| - \ln|-e|) \\ &= e \cdot (1 - 1) \\ &= 0\end{aligned}$$

Therefore, $\int_{-e}^{+e} \frac{edx}{x} = 0$ ∎

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► **Example 6.2.5.** Evaluate $\int_0^1 \frac{dx}{1-\sin(x)}$

Solution. In order to evaluate the integral, we simplify it as follows,

$$\begin{aligned}
\int_0^1 \frac{dx}{1-\sin(x)} &= \int_0^1 \frac{1}{(1-\sin(x))} \cdot \frac{(1+\sin(x))}{(1+\sin(x))} dx \\
&= \int_0^1 \frac{1+\sin(x)}{1-\sin^2(x)} dx \\
&= \int_0^1 \frac{1+\sin(x)}{\cos^2(x)} dx \\
&= \int_0^1 \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)} \right) dx \\
&= \int_0^1 \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \right) dx \\
&= \int_0^1 \left(\sec^2(x) + \tan(x) \cdot \sec(x) \right) dx \\
&= \int_0^1 \sec^2(x) dx + \int_0^1 \tan(x) \sec(x) dx
\end{aligned}$$

Now, we know, from the list of indefinite integral, that

$$\int \sec^2(x) dx = \tan(x) + C \qquad \text{and} \qquad \int \sec(x) \tan(x) dx = \sec(x) + C$$

Therefore, we write the following,

$$\begin{aligned}
\int_0^1 \sec^2(x) dx + \int_0^1 \tan(x) \sec(x) dx &= \Big[\tan(x)\Big]_0^1 + \Big[\sec(x)\Big]_0^1 \\
&= \Big[\tan(1) - \tan(0)\Big] + \Big[\sec(1) - \sec(0)\Big] \\
&\approx 2.40
\end{aligned}$$

Therefore, $\int_0^1 \frac{dx}{1-\sin(x)} \approx 2.40$ ∎

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———————The End of Section 6.2
The Fundamental Theorem of Calculus (FTC)

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Chapter 7

Methods of Integration

The methods of "integration" discussed in this chapter are more accurately be said as the methods of "antidifferentiation." This is because, as outlined in the previous chapter, the Fundamental Theorem of Calculus shows that any definite integral can typically be evaluated using antiderivatives, eliminating the need for the limit of approximations. From our experience, we can tell that using the limit of an approximation to evaluate a definite integral can be as impractical as to be totally impossible due to its complexity for complex functions. Fortunately, though, the Fundamental Theorem provides a simpler path—antidifferentiation.

However, this approach has its own limitations. So far, we only know the antiderivatives for a limited set of functions, specifically those whose derivatives we've encountered before. This is far from sufficient. For example, what is the antiderivative of $f(x) = \sec(x)$? In other words, what is that function $F(x)$ such that when it is differentiated, it equals $f(x)$? As of now, we don't have an answer, which means we are unable to evaluate any definite integral involving $f(x) = \sec(x)$. This presents a fundamental challenge.

Nevertheless, to address such challenges, several techniques have been developed to integrate—or more accurately, to antidifferentiate—functions, and this chapter will explore those methods, starting with the integration by substitution.

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§ 7.1 Integration by Substitution

Integration by substitution, or commonly known as "u-substitution," is fundamentally about simplifying an indefinite integral (i.e., the collection/family of antiderivatives) by changing its variable and then solving it. This method allows us to transform a complicated integral into a simpler one by introducing a new variable that makes it easier to solve.

Conceptually, this method can be thought of as a "backward rule" for the Chain Rule in differentiation. In differentiation, the Chain Rule allows us to differentiate a composite function by breaking it down into its constituent parts. For example, if $y = f(g(x))$, then the Chain rule states that the derivative of y with respect to x is $f'(g(x)) \cdot g'(x)$. Now, if we were to integrate the function $f'(g(x)) \cdot g'(x)$, the substitution rule allows us to simplify the integral by changing its variables.

In order to understand this method, consider a composite function $F(u)$, where $u = g(x)$ and that $F(u)$ is an antiderivative of some function $f(u)$, that is, $F'(u) = f(u)$. Now, differentiating $F(u) = F(g(x))$, we get,

$$\frac{d}{dx}F\left(g\left(x\right)\right) = F'\left(g\left(x\right)\right) \cdot g'\left(x\right)$$

Since $F'(u) = f(u)$, where $u = g(x)$, therefore,

$$F'\left(g\left(x\right)\right) \cdot g'\left(x\right) = f\left(g\left(x\right)\right) \cdot g'\left(x\right)$$
$$\therefore \frac{d}{dx}F\left(g\left(x\right)\right) = f\left(g\left(x\right)\right) \cdot g'\left(x\right)$$

This implies that $F\left(g\left(x\right)\right)$ is an antiderivative of $f\left(g\left(x\right)\right)g'\left(x\right)$. In notation, we write the following to denote the collection of all such antiderivatives.

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = F\left(g\left(x\right)\right) + C$$

Now, since $u = g(x)$, therefore $\dfrac{du}{dx} = g'(x)$ or in the differential form:

$$du = g'(x)dx$$

This suggests that $\displaystyle\int f\left(g\left(x\right)\right)g'\left(x\right)dx = F\left(g\left(x\right)\right)+C$ can be expressed in terms of the variable u by substituting $u = g(x)$ and $du = g(x)dx$, as follows:

$$\int f\underbrace{\left(g\left(x\right)\right)}_{u}\underbrace{g'\left(x\right)dx}_{du} = F\underbrace{\left(g\left(x\right)\right)}_{u}+C$$

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$$\int f(u)\,du = F(u) + C$$

By doing so, we can transform a complex integral $\int f(g(x))g'(x)\,dx = F(g(x)) + C$ with respect to x, into a simpler integral $\int f(u)\,du = F(u) + C$ with respect to a new variable u. This technique is useful because the new integral with respect to u is often easier to solve than the original integral with respect to x. Therefore, we have the following theorem.

Theorem 7.1.1. (Integration by Substitution)
Let $u = g(x)$ be a function such that $g'(x)$ is continuous over an interval, and let $f(u)$ be a continuous function over the corresponding range of $g(x)$. Then, if $F(u)$ is an antiderivative of $f(u)$, then

$$\begin{aligned}\int f(g(x))\,g'(x)\,dx &= \int f(u)\,du \\ &= F(u) + C \\ &= F(g(x)) + C\end{aligned}$$

The theorem above provides a method to simplify complex integrals involving composite functions. By introducing a new variable, we can transform these integrals into a simpler form that is easier to evaluate with respect to that new variable.

The challenge in using the Substitution Rule is selecting a suitable substitution for u. The key idea behind substitution is to simplify the integral, therefore, ideally, u should be a function whose differential (or a part of its differential) is present in the integrand. If this isn't possible, consider choosing u to be a more complex part of the integrand, such as the inner function of a composite function.

The bottom line is that finding the correct substitution often requires some trial and error. It's common to make an incorrect guess, so if the first choice doesn't work, simply try a different one.

To demonstrate the method, a few examples are provided below.

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► **Example 7.1.1.** Evaluate $\int (3x+4)^4 dx$

Solution. In order to evaluate this integral, let the variable $u = 3x + 4$, so that,

$$\begin{aligned} \frac{du}{dx} &= 3 \\ du &= 3dx \qquad \text{in differential form} \\ \therefore dx &= \frac{1}{3}du \end{aligned}$$

This implies that,

$$\begin{aligned} \int (3x+4)^4 dx &= \frac{1}{3} \cdot \int u^4 du \\ &= \frac{1}{3} \cdot \frac{u^5}{5} + C \\ &= \frac{u^5}{15} + C \end{aligned}$$

Since, $u = 3x + 4$, therefore,

$$\begin{aligned} \int (3x+4)^4 dx &= \frac{u^5}{15} + C \\ &= \frac{(3x+4)^5}{15} + C \end{aligned}$$

Therefore, $\int (3x+4)^4 dx = \frac{(3x+4)^5}{15} + C$ ∎

► **Example 7.1.2.** Evaluate $\int x\sqrt{x^2+2}dx$

Solution. In order to solve this integral, let the variable $u = \sqrt{x^2+2}$, so that,

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}\left(x^2+2\right) \\ &= 2x \\ du &= 2xdx \\ \therefore dx &= \frac{1}{2x}du \end{aligned}$$

This implies that,

$$\int x\sqrt{x^2+2}dx = \int x\sqrt{u}\frac{1}{2x}du$$

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$$
\begin{aligned}
&= x\frac{1}{2x} \cdot \int u^{\frac{1}{2}} du \\
&= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{2u^{\frac{3}{2}}}{6} + C \\
&= \frac{u^{\frac{3}{2}}}{3} + C \\
&= \frac{\left(x^2+2\right)^{\frac{3}{2}}}{3} + C
\end{aligned}
$$

Therefore, $\int x\sqrt{x^2+2}dx = \frac{\left(x^2+2\right)^{\frac{3}{2}}}{3} + C$ ∎

► **Example 7.1.3.** Evaluate $\int \frac{1}{x\ln(x)}dx$

Solution. In order to solve this integral, let the variable $u = \ln(x)$, so that,

$$
\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\ln(x) \\
&= \frac{1}{x} \\
\therefore du &= \frac{1}{x}dx
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\int \frac{1}{x\ln(x)}dx &= \int \underbrace{\frac{1}{\ln(x)}}_{u} \cdot \underbrace{\frac{1}{x}dx}_{du} \\
&= \int \frac{1}{u}du \\
&= \ln|u| + C \\
&= \ln|\ln(x)| + C
\end{aligned}
$$

Therefore, $\int \frac{1}{x\ln(x)}dx = \ln|\ln(x)| + C$ ∎

Sometimes the choice of u works for more than one substitution. For example, consider the following example.

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► **Example 7.1.4.** Evaluate $\int \frac{x}{\sqrt{x+1}}dx$

Solution. The integral above can be evaluated in multiple ways using the substitution rule. This is because the choice of u is not apparently visible. Whether we choose $u = x + 1$ or $u = \sqrt{x} + 1$, both yield the same answer.

Choosing $u = x + 1$: if $u = x + 1$, then $x = u - 1$, and

$$\begin{aligned} \frac{du}{dx} &= 1 \\ \therefore du &= dx \end{aligned}$$

This implies that,

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}}dx &= \int \frac{u-1}{\sqrt{u}}du \\ &= \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du \\ &= \int \frac{u}{\sqrt{u}}du - \int \frac{1}{\sqrt{u}}du \\ &= \int \sqrt{u}du - \int \frac{1}{\sqrt{u}}du \\ &= \int u^{\frac{1}{2}}du - \int u^{-\frac{1}{2}}du \\ &= \int u^{\frac{1}{2}}du - \int u^{-\frac{1}{2}}du \\ &= \frac{3}{2}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{3}{2}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C \end{aligned}$$

Choosing $u = \sqrt{x+1}$: if $u = \sqrt{x+1}$, then $u^2 = x + 1$ or $x = u^2 - 1$, and

$$\begin{aligned} \frac{d}{du}x &= \frac{d}{du}u^2 - \frac{d}{du}1 \\ \frac{dx}{du} &= 2u \\ \therefore dx &= 2udu \end{aligned}$$

This implies that,

$$\int \frac{x}{\sqrt{x+1}}dx = \int \frac{u^2-1}{u}2udu$$

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$$
\begin{aligned}
&= \int \frac{u^2-1}{u} 2u du \\
&= 2 \cdot \int \left(u^2-1\right) du \\
&= 2 \cdot \left(\int u^2 du - \int du\right) \\
&= 2 \cdot \left(\frac{u^3}{3} - u\right) + C \\
&= \frac{2u^3}{3} - 2u + C \\
&= \frac{2\left(\sqrt{x+1}\right)^3}{3} - 2\sqrt{x+1} + C \\
&= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + C
\end{aligned}
$$

Which yields the same result. Therefore, $\int \frac{x}{\sqrt{x+1}} dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + C$ ∎

Using the integration by substitution, a few formulas for integration can be derived as follows.

► **Example 7.1.5.** Evaluate $\int (ax+b)^n dx$

Solution. In order to evaluate the integral, we let,

$$u = ax + b \qquad \Longrightarrow \qquad du = a dx$$

This implies that $dx = \frac{du}{a}$. Thus, we write,

$$
\begin{aligned}
\int (ax+b)^n dx &= \int u^n \left(\frac{du}{a}\right) \\
&= \frac{1}{a} \cdot \int u^n du \\
&= \frac{1}{a} \cdot \frac{u^{n+1}}{n+1} + C \\
&= \frac{u^{n+1}}{a\,(n+1)} + C
\end{aligned}
$$

Since $u = ax + b$, therefore

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C$$

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Therefore, $\displaystyle\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C$ ∎

Remark 7.1.1.

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C \qquad n \neq 1$$

► **Example 7.1.6.** Evaluate $\displaystyle\int \sin(ax+b)dx$

Solution. In order to evaluate the integral, we let,

$$u = ax+b \qquad \Longrightarrow \qquad du = adx$$

This implies that $dx = \dfrac{du}{a}$. Thus, we write,

$$\begin{aligned}\int \sin(ax+b)\,dx &= \int \sin(u)\left(\frac{du}{a}\right)\\ &= \frac{1}{a}\cdot\int \sin(u)\,du\\ &= \frac{1}{a}\cdot(-\cos(u))\\ &= -\frac{1}{a}\cos(u)\end{aligned}$$

Since $u = ax+b$, therefore

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)$$

Therefore, $\displaystyle\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$ ∎

Remark 7.1.2.

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$$

► **Example 7.1.7.** Evaluate $\displaystyle\int \cos(ax+b)\,dx$

Solution. In order to evaluate the integral, we let,

$$u = ax+b \qquad \Longrightarrow \qquad du = adx$$

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This implies that $dx = \dfrac{du}{a}$. Thus, we write,

$$\begin{aligned}\int \cos(ax+b)\,dx &= \int \cos(u)\left(\frac{du}{a}\right)\\ &= \frac{1}{a}\cdot\int \cos(u)\,du\\ &= \frac{1}{a}\cdot \sin(u) + C\end{aligned}$$

Since $u = ax + b$, therefore

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

Therefore, $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$ ∎

Remark 7.1.3.

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

► **Example 7.1.8.** Evaluate $\displaystyle\int \tan(ax+b)\,dx$

Solution. In order to evaluate the integral, we let,

$$u = ax + b \qquad \Longrightarrow \qquad du = adx$$

This implies that $dx = \dfrac{du}{a}$. Thus, we write,

$$\begin{aligned}\int \tan(ax+b)\,dx &= \int \tan(u)\left(\frac{du}{a}\right)\\ &= \frac{1}{a}\cdot\int \tan(u)\,du\\ &= \frac{1}{a}\cdot\int \frac{\sin(u)}{\cos(u)}du\end{aligned}$$

Now, to solve the integral $\displaystyle\int \frac{\sin(u)}{\cos(u)}du$, we apply the substitution rule once again. That is, we let

$$v = \cos(u) \qquad \Longrightarrow \qquad dv = -\sin(u)du$$

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Thus, we write,

$$\begin{aligned}\int \frac{\sin(u)}{\cos(u)}du &= \int \frac{1}{v}(-dv)\\ &= -\int \frac{1}{v}dv\\ &= -\ln|v| + C\end{aligned}$$

Since $v = \cos(u)$ and $u = ax + b$, therefore, $v = \cos(ax + b)$. As a result, we write,

$$\begin{aligned}\int \tan(ax+b)\,dx &= \frac{1}{a}\cdot\int \frac{\sin(u)}{\cos(u)}du\\ &= \frac{1}{a}\cdot(-\ln|v| + C)\\ &= -\frac{1}{a}\ln|\cos(ax+b)| + C\\ &= -\frac{1}{a}\ln\left|\frac{1}{\sec(ax+b)}\right| + C\\ &= -\frac{1}{a}\left(\cancelto{0}{\ln|1|} - \ln|\sec(ax+b)|\right) + C\\ &= \frac{1}{a}\ln|\sec(ax+b)| + C\end{aligned}$$

Therefore, $\int \tan(ax+b)\,dx = -\frac{1}{a}\ln|\cos(ax+b)| + C = \frac{1}{a}\ln|\sec(ax+b)| + C$ ∎

Remark 7.1.4.

$$\int \tan(ax+b)\,dx = -\frac{1}{a}\ln|\cos(ax+b)| + C = \frac{1}{a}\ln|\sec(ax+b)| + C$$

► **Example 7.1.9.** Evaluate $\int \cot(ax+b)\,dx$

Solution. In order to evaluate the integral, we let,

$$u = ax + b \qquad \Longrightarrow \qquad du = a\,dx$$

This implies that $dx = \frac{du}{a}$. Thus, we write,

$$\int \cot(ax+b)\,dx = \int \cot(u)\left(\frac{du}{a}\right)$$

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$$= \frac{1}{a} \cdot \int \cot(u)\, du$$
$$= \frac{1}{a} \cdot \int \frac{\cos(u)}{\sin(u)} du$$

Now, to solve the integral $\int \frac{\cos(u)}{\sin(u)} du$, we apply the substitution rule once more. That is, we let,

$$v = \sin(u) \qquad \Longrightarrow \qquad dv = \cos(u) du$$

Thus, we write,

$$\int \frac{\cos(u)}{\sin(u)} du = \int \frac{1}{v} dv$$
$$= \ln|v| + C$$

Since $v = \sin(u)$ and $u = ax + b$, therefore, $v = \sin(ax + b)$. As a result, we write,

$$\begin{aligned}
\int \cot(ax+b)\, dx &= \frac{1}{a} \cdot \int \frac{\cos(u)}{\sin(u)} du \\
&= \frac{1}{a} \cdot (\ln|v| + C) \\
&= \frac{1}{a} \ln|\sin(ax+b)| + C \\
&= \frac{1}{a} \ln\left|\frac{1}{\csc(ax+b)}\right| + C \\
&= \frac{1}{a}\left(\cancelto{0}{\ln|1|} - \ln|\csc(ax+b)|\right) + C \\
&= -\frac{1}{a} \ln|\csc(ax+b)| + C
\end{aligned}$$

Therefore, $\int \cot(ax+b)\, dx = \frac{1}{a} \ln|\sin(ax+b)| + C = -\frac{1}{a} \ln|\csc(ax+b)| + C$ ∎

Remark 7.1.5.

$$\int \cot(ax+b)\, dx = \frac{1}{a} \ln|\sin(ax+b)| + C = -\frac{1}{a} \ln|\csc(ax+b)| + C$$

► **Example 7.1.10.** Evaluate $\int \sec(ax+b)\, dx$

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Solution. In order to evaluate the integral, we let,

$$u = ax + b \qquad \Longrightarrow \qquad du = adx$$

This implies that $dx = \dfrac{du}{a}$. Thus, we write,

$$\begin{aligned}\int \sec(ax+b)\,dx &= \int \sec(u)\left(\frac{du}{a}\right)\\ &= \frac{1}{a}\cdot\int \sec(u)\,du\\ &= \frac{1}{a}\cdot\int \sec(u)\cdot\frac{\tan(u)+\sec(u)}{\tan(u)+\sec(u)}du\\ &= \frac{1}{a}\cdot\int \frac{\sec(u)\tan(u)+\sec^2(u)}{\tan(u)+\sec(u)}du\end{aligned}$$

Now, to solve the integral $\displaystyle\int \frac{\sec(u)\tan(u)+\sec^2(u)}{\tan(u)+\sec(u)}du$, we apply the substitution rule once more. That is, we let,

$$v = \tan(u)+\sec(u) \qquad \Longrightarrow \qquad dv = \left(\sec(u)\tan(u)+\sec^2(u)\right)du$$

Thus, we write,

$$\begin{aligned}\int \frac{\sec(u)\tan(u)+\sec^2(u)}{\tan(u)+\sec(u)}du &= \int \frac{1}{v}dv\\ &= \ln|v|+C\end{aligned}$$

Since $v = \tan(u)+\sec(u)$ and $u = ax+b$, therefore, $v = \tan(ax+b)+\sec(ax+b)$. As a result, we write,

$$\begin{aligned}\int \sec(ax+b)\,dx &= \frac{1}{a}\cdot\int \frac{\sec(u)\tan(u)+\sec^2(u)}{\tan(u)+\sec(u)}du\\ &= \frac{1}{a}\cdot(\ln|v|+C)\\ &= \frac{1}{a}\ln|\tan(ax+b)+\sec(ax+b)|+C\end{aligned}$$

Therefore, $\displaystyle\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\tan(ax+b)+\sec(ax+b)|+C$ ∎

Remark 7.1.6.

$$\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\tan(ax+b)+\sec(ax+b)|+C$$

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► **Example 7.1.11.** Evaluate $\int \csc(ax+b)\,dx$

Solution. In order to evaluate the integral, we let,

$$u = ax + b \qquad \Longrightarrow \qquad du = a dx$$

This implies that $dx = \dfrac{du}{a}$. Thus, we write,

$$\begin{aligned}\int \csc(ax+b)\,dx &= \int \csc(u)\left(\frac{du}{a}\right)\\ &= \frac{1}{a}\cdot\int \csc(u)\,du\\ &= \frac{1}{a}\cdot\int \csc(u)\cdot\frac{\csc(u)+\cot(u)}{\csc(u)+\cot(u)}du\\ &= \frac{1}{a}\cdot\int \frac{\csc^2(u)+\cot(u)\csc(u)}{\csc(u)+\cot(u)}du\end{aligned}$$

Now, to solve the integral $\displaystyle\int \frac{\csc^2(u)+\cot(u)\csc(u)}{\csc(u)+\cot(u)}du$, we apply the substitution rule once more. That is, we let,

$$v = \csc(u)+\cot(u) \qquad \Longrightarrow \qquad dv = \big(-\csc^2(u)-\cot(u)\csc(u)\big)\,du$$

Thus, we write,

$$\begin{aligned}\int \frac{\csc^2(u)+\cot(u)\csc(u)}{\csc(u)+\cot(u)}du &= \int \frac{1}{v}(-dv)\\ &= -\int \frac{1}{v}dv\\ &= -\ln|v| + C\end{aligned}$$

Since $v = \csc(u)+\cot(u)$ and $u = ax+b$, therefore, $v = \csc(ax+b)+\cot(ax+b)$. As a result, we write,

$$\begin{aligned}\int \csc(ax+b)\,dx &= \frac{1}{a}\cdot\int \frac{\csc^2(u)+\cot(u)\csc(u)}{\csc(u)+\cot(u)}du\\ &= \frac{1}{a}\cdot(-\ln|v|+C)\\ &= -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C\end{aligned}$$

Therefore, $\displaystyle\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C$ ∎

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Remark 7.1.7.

$$\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C$$

► **Example 7.1.12.** Evaluate $\int \frac{f'(x)}{f(x)}dx$, where $f(x)\neq 0$ is any function.

Solution. In order to evaluate the integral, we let,

$$u = f(x) \qquad \Longrightarrow \qquad du = f'(x)dx$$

Thus, we write,

$$\begin{aligned}\int \frac{f'(x)}{f(x)}dx &= \int \frac{1}{u}du \\ &= \ln|f(x)|+C\end{aligned}$$

Therefore, $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$ ∎

Remark 7.1.8.

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$$

A few common integrals above are summarized below for a quick reference.

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C$
2. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$
3. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C$
4. $\int \tan(ax+b)\,dx = -\frac{1}{a}\ln|\sec(ax+b)|+C$
5. $\int \cot(ax+b)\,dx = \frac{1}{a}\ln|\sin(ax+b)|+C$
6. $\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\tan(ax+b)+\sec(ax+b)|+C$

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7. $\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C$
8. $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$

► **Example 7.1.13.** Evaluate $\int (5x+5)^5 dx$

Solution. We know that,

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

where, $n \neq -1$. Therefore,

$$\begin{aligned}\int (5x+5)^5 dx &= \frac{(5x+5)^{5+1}}{5(5+1)} + C \\ &= \frac{(5x+5)^6}{30} + C\end{aligned}$$

Therefore, $\int (5x+5)^5 dx = \frac{(5x+5)^6}{30} + C$ ∎

► **Example 7.1.14.** Evaluate $\int \frac{\cos(x)}{\sin(x)}dx$

Solution. We know that,

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$$

This implies that

$$\int \frac{\cos(x)}{\sin(x)}dx = \ln|\sin(x)|+C$$

Therefore, $\int \frac{\cos(x)}{\sin(x)}dx = \ln|\sin(x)|+C$ ∎

Chap. 7 / Sec. 7.1 / Subsec. 7.1.1 : Substitution Rule for Definite Integrals

The substitution rule can also be applied to definite integrals. For that, when changing variables from x to u, the limits of integration are also changed accordingly. That is, if $u = g(x)$, the original lower limit $x = a$ becomes $u = g(a)$, and

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the upper limit $x = b$ becomes $u = g(b)$. After substituting both the variable and the limits, we can complete the integration with respect to u. As a result, we have the following theorem.

Theorem 7.1.2. Let $u = g(x)$ be a differentiable function over an interval I, and let $f(u)$ be a continuous function over the rage of $g(x)$. If $F(u)$ is an antiderivative of $f(u)$ on I, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

► **Example 7.1.15.** Evaluate $\displaystyle\int_0^2 \frac{2x}{2+x^2}dx$

Solution. In order to solve the integral, let the variable $u = 2 + x^2$, then

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}(2 + x^2) \\ &= 2x \\ \therefore du &= 2xdx \end{aligned}$$

Now, since $u = 2 + x^2$, therefore,

lower limit $x = 0$	becomes	$u = 2 + 0^2 = 2$
upper limit $x = 2$	becomes	$u = 2 + 2^2 = 6$

This implies that,

$$\begin{aligned} \int_0^2 \frac{2x}{2+x^2}dx &= \int_0^2 \underbrace{\frac{1}{2+x^2}}_{u} \underbrace{2xdx}_{du} \\ &= \int_2^6 \frac{1}{u}du \\ &= \ln|u|\Big]_2^6 \\ &= \ln|6| - \ln|2| \\ &\approx 1.0986 \end{aligned}$$

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$$\int_0^2 \frac{2x}{2+x^2}dx \approx 1.0986$$ ∎

► **Example 7.1.16.** Evaluate $\int_e^{e^2} \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

Solution. In order to solve the integral, let the variable $u = e^x + e^{-x}$, then

$$\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\left(e^x + e^{-x}\right) \\
\frac{du}{dx} &= \frac{d}{dx}e^x + \frac{d}{dx}e^{-x} \\
\frac{du}{dx} &= e^x - e^{-x} \\
du &= \left(e^x - e^{-x}\right)dx \qquad \text{differential form}
\end{aligned}$$

Now, since $u = e^x + e^{-x}$, therefore,

$$\begin{array}{ccc}
\text{lower limit } x = e & \text{becomes} & u = e^e + e^{-e} \\
\text{upper limit } x = e^2 & \text{becomes} & u = e^{e^2} + e^{-e^2}
\end{array}$$

This implies that,

$$\begin{aligned}
\int_e^{e^2} \frac{e^x - e^{-x}}{e^x + e^{-x}}dx &= \int_e^{e^2} \underbrace{\frac{1}{e^x + e^{-x}}}_{u}\underbrace{\left(e^x - e^{-x}\right)dx}_{du} \\
&= \int_{e^e+e^{-e}}^{e^{e^2}+e^{-e^2}} \frac{1}{u}du \\
&= \Big[\ln|u|\Big]_{e^e+e^{-e}}^{e^{e^2}+e^{-e^2}} \\
&= \ln\left|e^{e^2} + e^{-e^2}\right| - \ln\left|e^e + e^{-e}\right| \\
&\approx 4.666
\end{aligned}$$

Therefore, $\int_e^{e^2} \frac{e^x - e^{-x}}{e^x + e^{-x}}dx \approx 4.666$ ∎

THE END OF SECTION 7.1

Integration by Substitution

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§ 7.2 Integration by Parts

Integration by parts is a technique used to solve integrals involving products of functions that are otherwise difficult to integrate directly. It can be thought of as a "backward rule" for the Product rule in differentiation. The core idea of this technique is to transform a difficult integral into the difference between a simple product and an integral that is easier to solve. The method is derived as follows.

Suppose, we're given $h(x) = f(x) \cdot g(x)$, then the product rule in differentiation states that,

$$\begin{aligned} \frac{d}{dx}h(x) &= \frac{d}{dx}\Big(f(x) \cdot g(x)\Big) \\ &= f\,(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f\,(x) \\ \therefore h'\,(x) &= f\,(x)\,g'\,(x) + g(x)f'\,(x) \end{aligned} \tag{1}$$

This implies—by the definition of an antiderivative—that $h(x)$ is an antiderivative of $f\,(x)\,g'\,(x) + g(x)f'\,(x)$, therefore, writing it in the integral form yields the following,

$$\begin{aligned} \int \Big(f\,(x)\,g'\,(x) + g(x)f'\,(x)\Big)dx &= h\,(x) \\ \int f\,(x)\,g'\,(x)\,dx + \int g(x)f'\,(x)\,dx &= f\,(x)\,g\,(x) \\ \int f\,(x)\,g'\,(x)\,dx &= f\,(x)\,g\,(x) - \int g(x)f'\,(x)\,dx \end{aligned}$$

Now, if we let $u = f(x)$, then the differential of u yields $du = f'(x)dx$. Similarly, if $v = g(x)$, then the differential of v yields $dv = g'(x)dx$. This implies the following,

$$\int \underbrace{f\,(x)}_{u}\underbrace{g'\,(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\underbrace{f'\,(x)\,dx}_{du}$$

Therefore, we have the following,

$$\int u\,dv = uv - \int v\,du$$

The method is based on the idea that if the integral $\int udv$ is challenging to evaluate, it can be simplified by rewriting it as $uv - \int vdu$, where $\int vdu$ is easier to solve. Thus, we have the following theorem.

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Theorem 7.2.1. (Integration by Parts)
If u and v are differentiable functions of x, then

$$\int u\,dv = uv - \int v\,du$$

A few examples of the theorem above are provided below.

► **Example 7.2.1.** Evaluate $\int x\sin(x)dx$

Solution. Notice that the integral $\int \sin(x)dx$ is easier to solve, however, the integral $\int x\sin(x)dx$ is not. In order to solve such an integral, we use integration by parts, stating the following.

$$\int u\,dv = uv - \int v\,du$$

This necessitates us to write the integral $\int x\sin(x)dx$ in the form $\int udv$. That is to identify which part of the integral $\int x\sin(x)dx$ is u and which part is dv.

Now to choose u and dv, notice that if we choose $u = x$, then $dv = \sin(x)dx$ fits a basic integration formula. Therefore, we let the following,

$$\begin{aligned} u = x \quad &\Longrightarrow \quad du = dx \\ dv = \sin(x)dx \quad &\Longrightarrow \quad v = \int dv = \int \sin(x)dx = -\cos(x) \end{aligned}$$

As a result, by integration by parts, we write the following.

$$\begin{aligned} \int \underbrace{x}_{u}\underbrace{\sin(x)\,dx}_{dv} &= \underbrace{x}_{u}\cdot\underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v}\underbrace{dx}_{du} \\ &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) + C \end{aligned}$$

Therefore, $\int x\sin(x)dx = -x\cos(x) + \sin(x) + C$ ∎

► **Example 7.2.2.** Evaluate $\int x\cos(x)dx$

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Solution. In order to solve the integral above, we will use integration by parts, stating the following.

$$\int u\,dv = uv - \int v\,du$$

This necessitates us to write the integral $\int x\sin(x)dx$ in the form $\int udv$. That is, to identify which part of the integral is $\int x\cos(x)dx$ is u and which part is dv.

If we let $u = x$, then $dv = \sin(x)dx$ fits a basic integration formula. Therefore, we let the following,

$$\begin{aligned} u = x \qquad &\Longrightarrow \qquad du = dx \\ dv = \cos(x)dx \qquad &\Longrightarrow \qquad v = \int dv = \int \cos(x)dx = \sin(x) \end{aligned}$$

As a result, by integration by parts, we write the following.

$$\begin{aligned} \int \underbrace{x}_{u}\underbrace{\cos(x)\,dx}_{dv} &= \underbrace{x}_{u}\underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v}\underbrace{dx}_{du} \\ &= x\sin(x) - (-\cos(x)) + C \\ &= x\sin(x) + \cos(x) + C \end{aligned}$$

Therefore, $\int x\cos(x)dx = x\sin(x) + \cos(x) + C$ ∎

► **Example 7.2.3.** Evaluate $\int \ln(x)dx$

Solution. In order to solve this, we use integration by parts. Therefore, we let,

$$\begin{aligned} u = \ln(x) \qquad &\Longrightarrow \qquad du = \frac{1}{x}dx \\ dv = dx \qquad &\Longrightarrow \qquad v = \int dv = \int dx = x \end{aligned}$$

Therefore, by integration by parts, we write the following,

$$\begin{aligned} \int \underbrace{\ln(x)}_{u}\underbrace{dx}_{dv} &= \underbrace{\ln(x)}_{u}\cdot\underbrace{x}_{v} - \int \underbrace{x}_{v}\cdot\underbrace{\frac{1}{x}dx}_{du} \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + C \end{aligned}$$

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Therefore, $\int \ln(x)dx = x\ln(x) - x + C$ ∎

Remark 7.2.1.

$$\int \ln(x)dx = x\ln(x) - x + C$$

Integration by parts is a handy method, especially when dv is easy to integrate. However, picking the right u and dv isn't straightforward and often comes down to experience and practice. A useful technique is to select u such that it becomes simpler when differentiated, and dv such that it remains easy to integrate, as done before. However, this technique is somewhat vague. A more structured method is described below.

Chap. 7 / Sec. 7.2 / Subsec. 7.2.1 : The LIATE Rule

The LIATE rule helps determine which part of an integrand to choose as u when using integration by parts. The acronym LIATE stands for the following set of functions.

1. **L**ogarithmic Functions (e.g. $\ln(x)$, $\log_a(b)$, etc.)
2. **I**nverse Trigonometric Functions (e.g. $\sin^{-1}(x)$, $\cos^{-1}(x)$, etc.)
3. **A**lgebraic Functions (e.g. x^2, $\sqrt{x}$, etc.)
4. **T**rigonometric Functions (e.g. $\sin(x)$, $\cos(x)$, etc.)
5. **E**xponential Functions (e.g. e^x, 2^x, etc.)

According to this rule, u is chosen as the first function that appears in the LIATE order (Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential). The remaining part of the integrand becomes dv.

For example, to integrate the integral $\int xe^x dx$ using integration by parts, we use the LIATE rule. Since x (an Algebraic function) appears before e^x (an Exponential function) in the LIATE order, therefore we let $u = x$ and $dv = e^x dx$.

It's important to note that while this method isn't foolproof, it works well enough to be considered useful. Below are a few examples where this rule applies perfectly.

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► **Example 7.2.4.** Evaluate $\int xe^x dx$

Solution. Notice that the integrand is a product of an algebraic function x and an exponential function e^x. According to the LIATE rule, we choose,

$$\begin{aligned} u = x \qquad &\Longrightarrow \qquad du = dx \\ dv = e^x dx \qquad &\Longrightarrow \qquad v = \int dv = \int e^x dx = e^x \end{aligned}$$

Thus, we write the following,

$$\begin{aligned} \int xe^x dx &= \int u\, dv \\ &= uv - \int v\, du \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= (x-1)\, e^x + C \end{aligned}$$

Therefore, $\int xe^x dx = (x-1)\, e^x + C$ ∎

Sometimes, evaluating an integral requires using integration by parts multiple times. This repeated process is necessary when the resulting integral after the first application can still be integrated using integration by parts. By repeatedly applying this method, we can gradually simplify the integral until we find a solution. For example, consider the following integral.

► **Example 7.2.5.** Evaluate $\int x^2 e^x dx$

Solution. By the LIATE rule, we choose,

$$\begin{aligned} u = x^2 \qquad &\Longrightarrow \qquad du = 2x dx \\ dv = e^x dx \qquad &\Longrightarrow \qquad v = \int dv = \int e^x dx = e^x \end{aligned}$$

Thus, we write the following,

$$\int e^x dx = \int u\, dv$$

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$$= uv - \int v\,du$$
$$= x^2e^x - \int e^x 2x\,dx$$

Now, notice that the integral $\int 2xe^x\,dx$ is of the form $\int f(x)g(x)dx$ such that it can be further integrated by integration by parts. In order to do that, we choose, by the LIATE rule, the following.

$$u = 2x \qquad \Longrightarrow \qquad du = 2dx$$
$$dv = e^x dx \qquad \Longrightarrow \qquad v = \int dv = \int e^x dx = e^x$$

Thus, we write the following,

$$\int e^x 2x\,dx = \int u\,dv$$
$$= uv - \int v\,du$$
$$= 2xe^x - \int e^x 2dx$$
$$= 2xe^x - 2\int e^x dx$$
$$= 2xe^x - 2e^x$$

This implies that

$$\int x^2e^x dx = x^2e^x - 2xe^x - 2e^x + C$$
$$= (x^2 - 2x - 2)e^x + C$$

Therefore, $\int x^2e^x dx = (x^2 - 2x - 2)e^x + C$ ∎

► **Example 7.2.6.** Evaluate $\int x^2 \sin(x)dx$

Solution. By the LIATE rule, we choose,

$$u = x^2 \qquad \Longrightarrow \qquad du = 2x\,dx$$
$$dv = \sin(x)dx \qquad \Longrightarrow \qquad v = \int dv = \int \sin(x)\,dx = -\cos(x)$$

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Thus, we write the following,

$$\begin{aligned}\int x^2 \sin(x)dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x^2(-\cos(x)) - \int (-\cos(x))2x\,dx \\ &= -x^2\cos(x) + 2\int x\cos(x)\,dx\end{aligned}$$

Now, notice that the integral $\int x\cos(x)\,dx$ is of the form $\int f(x)g(x)dx$ such that it can be further integrated by integration by parts. In order to do that, we choose, by the LIATE rule, the following.

$$\begin{aligned} u = x \qquad &\implies \qquad du = dx \\ dv = \cos(x)dx \qquad &\implies \qquad v = \int dv = \int \cos(x)dx = \sin(x)\end{aligned}$$

Thus, we write the following,

$$\begin{aligned}\int x\cos(x)dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) - (-\cos(x)) \\ &= x\sin(x) + \cos(x)\end{aligned}$$

This implies the following,

$$\begin{aligned}\int x^2\sin(x)dx &= -x^2\cos(x) + 2\left(x\sin(x) + \cos(x)\right) \\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\end{aligned}$$

Therefore, $\int x^2\sin(x)dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$ ∎

Chap. 7 / Sec. 7.2 / Subsec. 7.2.2 : Integration by Parts for Definite Integrals

Integration by parts can also be applied to definite integrals as stated by the following theorem.

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Theorem 7.2.2. Let u and v be differentiable functions of x, then

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)dx$$

A few examples are provided below.

► **Example 7.2.7.** Evaluate $\int_1^2 \ln(x)dx$

Solution. In order to integrate, we choose the following,

$$\begin{aligned} u = \ln(x) \qquad &\Longrightarrow \qquad du = \frac{1}{x}dx \\ dv = dx \qquad &\Longrightarrow \qquad v = \int dv = \int dx = x \end{aligned}$$

Therefore, we write the following,

$$\begin{aligned} \int_1^2 \ln(x)dx &= \int_1^2 \underbrace{u(x)}_{u}\underbrace{v'(x)dx}_{dv} \\ &= \underbrace{u(x)}_{u}\underbrace{v(x)}_{v}\Big]_1^2 - \int_1^2 \underbrace{v(x)}_{v}\underbrace{u'(x)dx}_{du} \\ &= \ln(x)\cdot x\Big]_1^2 - \int_1^2 x\frac{1}{x}dx \\ &= x\cdot\ln(x)\Big]_1^2 - \int_1^2 dx \\ &= \Big(2\cdot\ln(2) - 1\cdot\ln(1)\Big) - (2-1) \\ &= 2\ln(2) - 1 \\ &\approx 0.386 \end{aligned}$$

Therefore, $\int_1^2 \ln(x)dx \approx 0.386$ ∎

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Chap. 7 / Sec. 7.2 / Subsec. 7.2.3 : Integration by Reduction Formulas

Reduction formulas are formulas that express an integral involving a power of a function in terms of an integral involving a lower power of that same function. By repeatedly applying these formulas, we can reduce a complex integral into simpler forms, eventually arriving at a solvable integral. For example, we can derive a reduction formula for the integral $\int \sin^n(x)dx$, by noting that $\sin^n(x)$ can be written as $\sin^{n-1}(x) \cdot \sin(x)$. This implies

$$\int \sin^n(x)\,dx = \int \sin^{n-1}(x)\sin(x)\,dx$$

Now, if we let

$$\begin{aligned} u = \sin^{n-1}(x) \qquad &\implies \qquad du = (n-1)\sin^{n-2}(x)\cos(x)dx \\ dv = \sin(x)dx \qquad &\implies \qquad v = \int dv = \int \sin(x)dx = -\cos(x) \end{aligned}$$

then we have the following,

$$\begin{aligned} \int udv =& uv - \int vdu \\ \int \sin^{n-1}(x)\sin(x)\,dx =& \sin^{n-1}(x)\cdot(-\cos(x)) \\ & - \int (-\cos(x))\cdot(n-1)\sin^{n-2}(x)\cos(x)\,dx \\ \int (\sin(x))^{n-1}(\sin(x))^1 dx =& -\sin^{n-1}(x)\cos(x) \\ & + (n-1)\int \cos^2(x)\cdot\sin^{n-2}(x)\,dx \\ \int (\sin(x))^{n-1+1} dx =& -\sin^{n-1}(x)\cos(x) \\ & + (n-1)\int \left(1-\sin^2(x)\right)\cdot\sin^{n-2}(x)\,dx \\ \int \sin^n(x)\,dx =& -\sin^{n-1}(x)\cos(x) \\ & + (n-1)\int \sin^{n-2}(x) - \sin^2(x)\sin^{n-2}(x)\,dx \\ \int \sin^n(x)\,dx =& -\sin^{n-1}(x)\cos(x) \\ & + (n-1)\int \sin^{n-2}(x) - \sin^n(x)\,dx \\ \int \sin^n(x)\,dx =& -\sin^{n-1}(x)\cos(x) \end{aligned}$$

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$$+ (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^{n}(x)\,dx$$

$$\int \sin^{n}(x)\,dx + (n-1)\int \sin^{n}(x)\,dx = -\sin^{n-1}(x)\cos(x)$$
$$+ (n-1)\int \sin^{n-2}(x)\,dx$$

$$n\int \sin^{n}(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$$

$$\int \sin^{n}(x)\,dx = \frac{-\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx}{n}$$

$$= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

This implies that

$$\int \sin^{n} dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

Similar formulas can be obtained for other trigonometric functions as well.

1. $\displaystyle\int \sin^{n}(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$
2. $\displaystyle\int \cos^{n}(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$
3. $\displaystyle\int \tan^{n}(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx, \quad n \neq 1$
4. $\displaystyle\int \cot^{n}(x)\,dx = -\frac{1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx, \quad n \neq 1$
5. $\displaystyle\int \sec^{n}(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) - \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx,$ $n \neq 1$
6. $\displaystyle\int \csc^{n}(x)\,dx = \frac{1}{n-1}\csc^{n-2}(x)\cot(x) - \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx,$ $n \neq 1$

———— The End of Section 7.2

Integration by Parts

§ 7.3 Trigonometric Integrals

So far, we have discussed various methods to integrate integrals that are relatively easy to solve. These methods have so far been instrumental, but when

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faced with integrals involving complicated functions such as trigonometric functions raised to any powers or in products, we may fall short of techniques to integrate them. For example, from our current knowledge, we can easily integrate the following types of integral,

$$\int e^x \sin(x) dx$$

The same, however, cannot be said for integrating the following types of integrals,

$$\int \sin^2(x) dx \qquad \text{or} \qquad \int \sin(x)\cos(x) dx$$

This is because neither the substitution rule nor integration by parts is immediately useful in this case, as the trigonometric expressions don't lend themselves to obvious substitutions or algebraic simplifications. This limitation becomes even more apparent when considering even complex examples, such as:

$$\int \sin^4(x)\cos^3(x) dx$$

These types of integrals fall under the category of trigonometric integrals, which require different strategies than the ones we have discussed so far. In this section, we aim to develop techniques for integrating such integrals. We will do so by manipulating trigonometric expressions using identities, reducing powers, and transforming products into more manageable forms.

Chap. 7 / Sec. 7.3 / Subsec. 7.3.1 : Integrals Involving Powers of $\sin(x)$ and $\cos(x)$

In order to integrate integrals of the form,

$$\int \sin^m(x) dx \qquad \text{and} \qquad \int \cos^n(x) dx$$

two strategies are primarily used based on whether the power is odd or even. For example, with even powers of $\sin(x)$ or $\cos(x)$, we use the half-angle formulas,

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

to reduce the powers of $\sin(x)$ or $\cos(x)$ in the integrand.

Similarly, with odd powers of $\sin(x)$ or $\cos(x)$, the best idea is to split off a single factor of $\sin(x)$ or $\cos(x)$ and to use the Pythagorean identities in a way that

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makes the substitution rule useful.

These techniques can best be explained through the following examples.

► **Example 7.3.1.** Evaluate $\int \cos^3(x)dx$

Solution. For odd powers, we use substitution by expressing the cosine function in terms of sine. For that, we rewrite $\cos^3(x)$ as the following,

$$\cos^3(x) = \cos^2(x) \cdot \cos(x)$$

Now, since $\cos^2(x) = 1 - \sin^2(x)$, therefore, the integral becomes,

$$\begin{aligned}\int \cos^3(x)dx &= \int \cos^2(x)\cos(x)dx \\ &= \int \left(1 - \sin^2(x)\right)\cos(x)dx\end{aligned}$$

Now, let $u = \sin(x)$, so, $du = \cos(x)dx$. Then,

$$\begin{aligned}\int \left(1 - \sin^2(x)\right)\cos(x)dx &= \int \left(1 - u^2\right)du \\ &= \int du - \int u^2 du \\ &= u - \frac{u^3}{3} + C \\ &= \sin(x) - \frac{\sin^3(x)}{3} + C\end{aligned}$$

Therefore, $\int \cos^3(x)dx = \sin(x) - \dfrac{\sin^3(x)}{3} + C$ ∎

► **Example 7.3.2.** Evaluate $\int \sin^4(x)dx$

Solution. For even powers, we express $\sin^4(x)$ as $(\sin^2(x))^2$, and then use the half-angle formula for $\sin^2(x)$. That is, since,

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

therefore, we write the following,

$$\int \sin^4(x)dx = \int \left(\sin^2(x)\right)^2 dx$$

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$$
\begin{aligned}
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\
&= \int \frac{(1-\cos(2x))^2}{4} dx \\
&= \int \frac{1^2 - 2\cos(2x) + \cos^2(2x)}{4} dx \\
&= \frac{1}{4} \cdot \int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx \\
&= \frac{1}{4} \cdot \left(\int dx - 2\int \cos(2x)dx + \int \cos^2(2x)dx\right)
\end{aligned}
$$

Now, to evaluate the integral $\int \cos^2(2x)dx$, we use the following half-angle formula for $\cos^2(2x)$, that is,

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

This implies that

$$
\begin{aligned}
\int \sin^4(x)dx &= \frac{1}{4} \cdot \left(\int dx - 2\int \cos(2x)dx + \int \cos^2(2x)dx\right) \\
&= \frac{1}{4} \cdot \left(\int dx - 2\int \cos(2x)dx + \int \frac{1+\cos(4x)}{2} dx\right) \\
&= \frac{1}{4} \cdot \left(\int dx - 2\int \cos(2x)dx + \frac{1}{2} \cdot \int \left(1+\cos(4x)\right) dx\right) \\
&= \frac{1}{4} \cdot \left(\int dx - 2\int \cos(2x)dx + \frac{1}{2} \cdot \left(\int dx + \int \cos(4x)dx\right)\right) \\
&= \frac{1}{4} \cdot \left(x - 2 \cdot \frac{\sin(2x)}{2} + \frac{1}{2} \cdot \left(x + \frac{\sin(4x)}{4}\right)\right) + C \\
&= \frac{1}{4} \cdot \left(x - \sin(2x) + \frac{1}{2}x + \frac{1}{2}\frac{\sin(4x)}{4}\right) + C \\
&= \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C
\end{aligned}
$$

Therefore, $\int \sin^4(x)dx = \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C$ ∎

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Chap. 7 / Sec. 7.3 / Subsec. 7.3.2 : Integrals Involving Product of Powers of $\sin(x)$ and $\cos(x)$

We now consider integrals involving products of $\sin(x)$ and $\cos(x)$ of the following form,

$$\int \sin^m(x)\cos^n(x)\,dx$$

where, m and n are positive integers.

In order to solve such integrals, we refer to the following steps.

1. If m in the integral $\int \sin^m(x)\cos^n(x)\,dx$ is odd, then
   (a) Split off a factor of $\sin(x)$
   (b) Use the identity $\sin^2(x) = 1 - \cos^2(x)$
   (c) Make the substitution $u = \cos(x)$ and $du = -\sin(x)dx$
2. If n in the integral $\int \sin^m(x)\cos^n(x)\,dx$ is odd, then
   (a) Split off a factor of $\cos(x)$
   (b) Use the identity $\cos^2(x) = 1 - \sin^2(x)$
   (c) Make the substitution $u = \sin(x)$ and $du = \cos(x)dx$
3. If both m, n in the integral $\int \sin^m(x)\cos^n(x)\,dx$ is even, then use the following identities to reduce the powers on $\sin(x)$ and $\cos(x)$.

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

► **Example 7.3.3.** Evaluate $\int \sin^4(x)\cos^3(x)\,dx$

Solution. To solve the integral, we follow the procedure above.

$$\begin{aligned}\int \sin^4(x)\cos^3(x)\,dx &= \int \sin^4(x)\cos^2(x)\cos(x)\,dx \\ &= \int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)\,dx\end{aligned}$$

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Now, let $u = \sin(x)$, then $du = \cos(x)dx$. Thus, we write the following,

$$\begin{aligned}\int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)\,dx &= \int u^4\left(1-u^2\right)du \\ &= \int \left(u^4-u^6\right)du \\ &= \int u^4du - \int u^6du \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^6(x)}{6} - \frac{\sin^8(x)}{8} + C\end{aligned}$$

Therefore, $\int \sin^4(x)\cos^3(x)\,dx = \frac{\sin^6(x)}{6} - \frac{\sin^8(x)}{8} + C$ ∎

► **Example 7.3.4.** Evaluate $\int \sin^3(x)\cos^4(x)\,dx$

Solution. In order to solve the integral, we follow the procedure above.

$$\begin{aligned}\int \sin^3(x)\cos^4(x)\,dx &= \int \cos^4(x)\sin^2(x)\sin(x)\,dx \\ &= \int \cos^4(x)\left(1-\cos^2(x)\right)\sin(x)\,dx\end{aligned}$$

Now, let $u = \cos(x)$, then $du = -\sin(x)dx$. Thus, we write the following,

$$\begin{aligned}\int \cos^4(x)\left(1-\cos^2(x)\right)\sin(x)\,dx &= \int u^4\left(1-u^2\right)(-du) \\ &= -\int u^4\left(1-u^2\right)du \\ &= -\int \left(u^4-u^6\right)du \\ &= -\int u^4du + \int u^6du \\ &= -\frac{u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C\end{aligned}$$

Therefore, $\int \sin^3(x)\cos^4(x)\,dx = \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C$ ∎

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Chap. 7 / Sec. 7.3 / Subsec. 7.3.3 : Integrals Involving Powers of $\tan(x)$ and $\sec(x)$

The idea behind integrating integrals involving powers of $\tan(x)$ and $\sec(x)$ is similar to that of integrals involving powers of $\sin(x)$ and $\cos(x)$ as discussed before. For example, consider integrating the following integral.

► **Example 7.3.5.** Evaluate $\int \sec^3(x)dx$

Solution. For odd powers, we use integration by parts by rewriting $\sec^3(x)$ as $\sec(x)\sec^2(x)$ and letting,

$$\begin{aligned} u = \sec(x) \quad &\implies \quad du = \sec(x)\tan(x)dx \\ dv = \sec^2(x)dx \quad &\implies \quad v = \int dv = \int \sec^2(x)dx = \tan(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sec^3(x)dx &= \int \sec(x)\sec^2(x)dx \\ &= \int udv \\ &= uv - \int vdu \\ &= \sec(x)\tan(x) - \int \tan(x)\cdot\sec(x)\tan(x)dx \\ &= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx \end{aligned}$$

Now, since $\tan^2(x) = \sec^2(x) - 1$, therefore,

$$\begin{aligned} \int \sec^3(x)dx &= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx \\ \int \sec^3(x)dx &= \sec(x)\tan(x) - \int \left(\sec^2(x) - 1\right)\sec(x)dx \\ \int \sec^3(x)dx &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \\ \int \sec^3(x)dx + \int \sec^3(x)dx &= \sec(x)\tan(x) + \int \sec(x)dx \\ 2\cdot\int \sec^3(x)dx &= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C \\ \int \sec^3(x)dx &= \frac{\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C}{2} \end{aligned}$$

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$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$$

Therefore, $\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$ ∎

► **Example 7.3.6.** Evaluate $\int \tan^4(x)dx$

Solution. For even powers, we rewrite $\tan^4(x)$ as $\tan^2(x)\cdot\tan^2(x)$ and use the following identity,

$$\tan^2(x) + \sec^2(x) = 1$$

This implies that,

$$\begin{aligned}
\int \tan^4(x)dx &= \int \tan^2(x)\cdot\tan^2(x)dx \\
&= \int \tan^2(x)\cdot\left(\sec^2(x)-1\right)dx \\
&= \int \left(\tan^2(x)\sec^2(x) - \tan^2(x)\right)dx \\
&= \int \tan^2(x)\sec^2(x)dx - \int \tan^2(x)dx \\
&= \int \tan^2(x)\sec^2(x)dx - \int \left(\sec^2(x)-1\right)dx \\
&= \int \tan^2(x)\sec^2(x)dx - \int \sec^2(x)dx + \int dx
\end{aligned}$$

Now, let $u = \tan(x)$, then $du = \sec^2(x)dx$ in the first integral. Therefore,

$$\begin{aligned}
\int \tan^4(x)dx &= \int \tan^2(x)\sec^2(x)dx - \int \sec^2(x)dx + \int dx \\
&= \int u^2 du - \int \sec^2(x)dx + \int dx \\
&= \frac{u^3}{3} - \int \sec^2(x)dx + \int dx \\
&= \frac{\tan^3(x)}{3} - \tan(x) + x + C
\end{aligned}$$

Therefore, $\int \tan^4(x)dx = \frac{\tan^3(x)}{3} - \tan(x) + x + C$ ∎

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Chap. 7 / Sec. 7.3 / Subsec. 7.3.4 : Integrals Involving Product of Powers of $\tan(x)$ and $\sec(x)$

We now consider integrals involving $\tan(x)$ and $\sec(x)$ of the following form,

$$\int \tan^m(x)\sec^n(x)dx$$

where, m and n are positive integers.

In order to solve such integrals, we refer to the following steps.

1. If m in the integral $\int \tan^m(x)\sec^n(x)\,dx$ is odd, then
   (a) Split off a factor of $\sec(x)\tan(x)$
   (b) Use the identity $\tan^2(x) = \sec^2(x) - 1$
   (c) Make the substitution $u = \sec(x)$ and $du = \sec(x)\tan(x)dx$
2. If n in the integral $\int \tan^m(x)\sec^n(x)\,dx$ is even, then
   (a) Split off a factor of $\sec^2(x)$
   (b) Use the identity $\sec^2(x) = \tan^2(x) + 1$
   (c) Make the substitution $u = \tan(x)$ and $du = \sec^2(x)dx$
3. If m is even and n is odd in the integral $\int \tan^m(x)\sec^n(x)\,dx$, then use the following identities to reduce the powers on $\sec(x)$.
   (a) $\tan^2(x) = \sec^2(x) - 1$
   (b) Use the reduction formula for powers of $\sec(x)$.

► **Example 7.3.7.** Evaluate $\int \tan^3(x)\sec^4(x)\,dx$

Solution. In order to solve the integral, we follow the procedure above.

$$\begin{aligned}\int \tan^3(x)\sec^4(x)\,dx &= \int \tan^3(x)\sec^2(x)\sec^2(x)\,dx \\ &= \int \tan^3(x)\left(\tan^2(x)+1\right)\sec^2(x)\,dx\end{aligned}$$

Now, let $u = \tan(x)$, then $du = \sec^2(x)dx$. Thus, we write the following,

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\sec^2(x)\sec^2(x)\,dx$$

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$$= \int \tan^3(x)\left(\tan^2(x)+1\right)\sec^2(x)\,dx$$

$$\begin{aligned}\int \tan^3(x)\left(\tan^2(x)+1\right)\sec^2(x)\,dx &= \int u^3\left(u^2+1\right)du\\ &= \int \left(u^5+u^3\right)du\\ &= \int u^5du + \int u^3du\\ &= \frac{u^6}{6}+\frac{u^4}{4}+C\\ &= \frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C\end{aligned}$$

Therefore, $\int \tan^3(x)\sec^4(x)\,dx = \frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C$ ∎

———————The End of Section 7.3
Trigonometric Integrals

§ 7.4 Trigonometric Substitution

Trigonometric substitution is a technique to simplify integrals that are otherwise difficult to solve. It is used in cases when the square root of a sum or difference of two squares is not easy to simplify (that is, of the form: $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, and $\sqrt{x^2-a^2}$), but the square root of a product of two squares is (that is, of the form: $\sqrt{A^2B^2}=|AB|$). For that reason, integrals such as the following

$$\int \sqrt{1+x^2}dx$$

is quite difficult to solve. Nevertheless, if we can replace $\sqrt{1+x^2}$ with a product of squares, then it becomes easier to solve.

The goal of this section is to transform the sum or difference of squares, that is, $\sqrt{a^2\pm x^2}$, and $\sqrt{x^2-a^2}$, into products of squares. It turns out that $\sqrt{a^2\pm x^2}$, and $\sqrt{x^2-a^2}$ can be transformed into products of squares by seemingly unexpected substitution: trigonometric substitution. Really, what do trigonometric functions have to do with the sum or difference of squares? This is what we see in the following subsections.

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Chap. 7 / Sec. 7.4 / Subsec. 7.4.1 : Integrals Involving $\sqrt{a^2 - x^2}$

Suppose, we're given an integral whose integrand involves the term $\sqrt{a^2 - x^2}$, where a is a positive constant. Now, notice what happens when we substitute $x = a\sin(\theta)$ in the expression $\sqrt{a^2 - x^2}$, as shown below.

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin(\theta))^2} \\ &= \sqrt{a^2 - a^2\sin^2(\theta)} \\ &= \sqrt{a^2\left(1 - \sin^2(\theta)\right)}\end{aligned}$$

It is known from the Pythagorean identity that $\sin^2(\theta) + \cos^2(\theta) = 1$, or, $\cos^2(\theta) = 1 - \sin^2(\theta)$. Therefore,

$$\begin{aligned}&= \sqrt{a^2 \underbrace{\left(1 - \sin^2(\theta)\right)}_{\cos^2(\theta)}} \\ &= \sqrt{a^2 \cdot \cos^2(\theta)} \\ &= |a \cdot \cos(\theta)| \\ &= a\cos(\theta) \qquad (1)\end{aligned}$$

This implies that the substitution $x = a\sin(\theta)$ transform the difficult $\sqrt{a^2 - x^2}$ into an easier product $a\cos(\theta)$. That is,

$$\sqrt{a^2 - x^2} = a\cos(\theta)$$

if we let $x = a\sin(\theta)$.

The substitution $x = a\sin(\theta)$ for $\sqrt{a^2 - x^2}$ works because, if we consider a right-angled triangle whose one side is x and the hypotenuse is a, the length of the other side becomes $\sqrt{a^2 - x^2}$ from the Pythagorean theorem, as shown below.

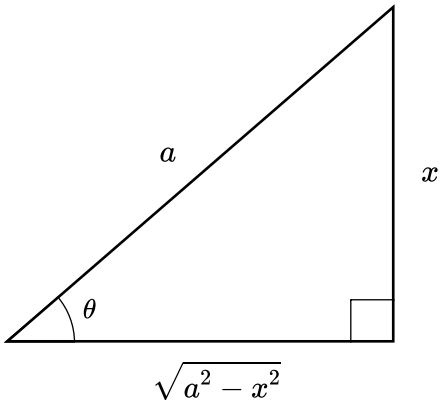

Figure 7.1

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This implies that,

$$\sin(\theta) = \frac{x}{a}$$
$$\therefore x = a\sin(\theta)$$

A few examples of the technique above are provided below.

► **Example 7.4.1.** Evaluate $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

Solution. Since $a = 2$ in the integral above, therefore to solve the integral, let $x = 2\sin(\theta)$, then $dx = 2\cos(\theta)d\theta$. This implies that,

$$\begin{aligned}
\int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{2\cos(\theta)d\theta}{\sqrt{4-(2\sin(\theta))^2}} \\
&= \int \frac{2\cos(\theta)}{\sqrt{4-(2\sin(\theta))^2}}d\theta \\
&= \int \frac{2\cos(\theta)}{\sqrt{4-4\sin^2(\theta)}}d\theta \\
&= \int \frac{2\cos(\theta)}{\sqrt{4\left(1-\sin^2(\theta)\right)}}d\theta \\
&= \int \frac{2\cos(\theta)}{\sqrt{4}\cdot\sqrt{\cos^2(\theta)}}d\theta \\
&= \int \frac{\cancel{2\cos(\theta)}}{\cancel{2\cos(\theta)}}d\theta \\
&= \int d\theta \\
&= \theta + C
\end{aligned}$$

Since $x = 2\sin(\theta)$, therefore, $\theta = \sin^{-1}\left(\frac{x}{2}\right)$. As a result,

$$\begin{aligned}
\int \frac{dx}{\sqrt{4-x^2}} &= \theta + C \\
&= \sin^{-1}\left(\frac{x}{2}\right) + C
\end{aligned}$$

Therefore, $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C$ ∎

► **Example 7.4.2.** Evaluate $\displaystyle\int \sqrt{9-x^2}dx$

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Solution. Since $a = 3$ in the integral above, therefore, let $x = 3\sin(\theta)$, then $dx = 3\cos(x)d\theta$. This implies the following,

$$\begin{aligned}
\int \sqrt{9-x^2}dx &= \int \sqrt{9-(3\sin(\theta))^2}3\cos(\theta)d\theta \\
&= \int \sqrt{9-9\sin^2(\theta)}3\cos(\theta)d\theta \\
&= \int \sqrt{9\left(1-\sin^2(\theta)\right)}3\cos(\theta)d\theta \\
&= \int \sqrt{9\left(\cos^2(\theta)\right)}3\cos(\theta)d\theta \\
&= \int \sqrt{9\left(\cos^2(\theta)\right)}3\cos(\theta)d\theta \\
&= \int \sqrt{9}\cdot\sqrt{\cos^2(\theta)}3\cos(\theta)d\theta \\
&= \int 3\cos(\theta)\cdot 3\cos(\theta)d\theta \\
&= 9\int \cos^2(\theta)d\theta
\end{aligned}$$

Since it is known, from the double-angle identity, that $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$, therefore,

$$\begin{aligned}
9\int \cos^2(\theta)d\theta &= 9\int \frac{1+\cos(2\theta)}{2}d\theta \\
&= \frac{9}{2}\int \left(1+\cos(2\theta)\right)d\theta \\
&= \frac{9}{2}\left(\int d\theta + \int \cos(2\theta)d\theta\right) \\
&= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2} + C\right)
\end{aligned}$$

Since it is known, from the double-angle identity, that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, therefore,

$$\begin{aligned}
9\int \cos^2(\theta)d\theta &= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2} + C\right) \\
&= \frac{9}{2}\left(\theta + \frac{2\sin(\theta)\cos(\theta)}{2} + C\right)
\end{aligned}$$

Since $x = 3\sin(\theta)$, therefore $\theta = \sin^{-1}\left(\dfrac{x}{3}\right)$. Thus,

$$\int \sqrt{9-x^2}dx = \frac{9}{2}\left(\theta + \frac{2\sin(\theta)\cos(\theta)}{2} + C\right)$$

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$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{2\sin\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)}{2} + C\right)$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \sin\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) + C\right)$$

Since $\sin(\sin^{-1}(x)) = x$ and $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, therefore,

$$\begin{aligned}\int \sqrt{9-x^2}dx &= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \sin\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) + C\right)\\ &= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\sqrt{1-\left(\frac{x}{3}\right)^2} + C\right)\\ &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{3x}{2}\sqrt{1-\left(\frac{x}{3}\right)^2} + C\end{aligned}$$

Therefore, $\int \sqrt{9-x^2}dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{3x}{2}\sqrt{1-\left(\frac{x}{3}\right)^2} + C$ ∎

We now generalize the above integrals into formulas.

► **Example 7.4.3.** Evaluate $\int \sqrt{a^2-x^2}dx$

Solution. To evaluate the integral, we let $x = a\sin(\theta)$. The the differential of x becomes $dx = a\cos(\theta)d\theta$. Therefore,

$$\begin{aligned}\int \sqrt{a^2-x^2}dx &= \int \sqrt{a^2-(a\sin(\theta))^2}a\cos(\theta)\,d\theta\\ &= \int \sqrt{a^2-a^2\sin^2(\theta)}a\cos(\theta)\,d\theta\\ &= \int \sqrt{a^2\left(1-\sin^2(\theta)\right)}a\cos(\theta)\,d\theta\\ &= \int \sqrt{a^2}\cdot\sqrt{\left(1-\sin^2(\theta)\right)}a\cos(\theta)\,d\theta\\ &= \int a\cdot\sqrt{\cos^2(\theta)}a\cos(\theta)\,d\theta\\ &= a\int \cos(\theta)\cos(\theta)\,d\theta\\ &= a^2\int \cos^2(\theta)\,d\theta \qquad \text{(E1)}\end{aligned}$$

To deal with the integrand of the above integral, we make use of the half-angle identity, stating that,

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

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Therefore, continuing on with equation (E1), we get,

$$\begin{aligned}
a^2 \int \cos^2(\theta)\, d\theta &= a^2 \int \left(\frac{1+\cos(2\theta)}{2} \right) d\theta \\
&= a^2 \int \left(\frac{1}{2} + \frac{\cos(2\theta)}{2} \right) d\theta \\
&= a^2 \int \frac{1}{2} d\theta + a^2 \int \frac{\cos(2\theta)}{2} d\theta \\
&= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos(2\theta)\, d\theta \\
&= \frac{a^2}{2} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C \\
&= \frac{a^2}{2} \left(\theta + \frac{2\sin(\theta)\cos(\theta)}{2} \right) + C \\
&= \frac{a^2}{2} \left(\theta + \sin(\theta)\cos(\theta) \right) + C \qquad \text{(E2)}
\end{aligned}$$

Now, from the Pythagorean identity, it is known that,

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\cos^2(\theta) &= 1 - \sin^2(\theta) \\
\cos(\theta) &= \sqrt{1 - \sin^2(\theta)} \qquad \text{(E3)}
\end{aligned}$$

Substituting equation (E3) into (E2) yields the following.

$$\frac{a^2}{2} \left(\theta + \sin(\theta)\cos(\theta) \right) + C = \frac{a^2}{2} \left(\theta + \sin(\theta) \sqrt{1 - \sin^2(\theta)} \right) + C \qquad \text{(E4)}$$

Now, since $x = a\sin(\theta)$, therefore,

$$\begin{aligned}
x &= a\sin(\theta) \\
\frac{x}{a} &= \sin(\theta) \\
\therefore \theta &= \sin^{-1}\left(\frac{x}{a} \right)
\end{aligned}$$

Substituting $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ into equation (E4) yields the following,

$$\begin{aligned}
&= \frac{a^2}{2} \left(\theta + \sin(\theta) \sqrt{1 - \sin^2(\theta)} \right) + C \\
&= \frac{a^2}{2} \left(\sin^{-1}\left(\frac{x}{a}\right) + \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right) \sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x}{a}\right)\right)} \right) + C \\
&= \frac{a^2}{2} \left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a} \cdot \sqrt{1 - \left(\frac{x}{a}\right)^2} \right) + C
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}\right) + C \\
&= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + C \\
&= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}a\sqrt{\frac{1}{a^2}}\cdot\sqrt{a^2-x^2} + C \\
&= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\frac{a}{\sqrt{a^2}}\cdot\sqrt{a^2-x^2} + C \\
&= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C
\end{aligned}
$$

This suggests the following,

$$\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

∎

Remark 7.4.1.

$$\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

► **Example 7.4.4.** Evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$

Solution. To evaluate the integral, we let $x = a\sin(\theta)$. Then, the differential $dx = a\cos(\theta)d\theta$. Therefore,

$$
\begin{aligned}
\int \frac{dx}{\sqrt{a^2-x^2}} &= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2-(a\sin(\theta))^2}} \\
&= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2-a^2\sin^2(\theta)}} \\
&= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2\left(1-\sin^2(\theta)\right)}} \\
&= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2}\cdot\sqrt{\left(1-\sin^2(\theta)\right)}} \\
&= \int \frac{a\cos(\theta)\,d\theta}{a\sqrt{\left(1-\sin^2(\theta)\right)}}
\end{aligned}
$$

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$$
\begin{aligned}
&= \int \frac{a\cos(\theta)\,d\theta}{a\sqrt{\cos^2(\theta)}} \\
&= \int \frac{a\cos(\theta)\,d\theta}{a\cos(\theta)} \\
&= \int d\theta \\
&= \theta + C
\end{aligned} \tag{E1}
$$

Now, since $x = a\sin(\theta)$, therefore,

$$
\begin{aligned}
x &= a\sin(\theta) \\
\frac{x}{a} &= \sin(\theta) \\
\therefore \theta &= \sin^{-1}\left(\frac{x}{a}\right)
\end{aligned}
$$

Now, substituting $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ into equation (E1), we have the following,

$$\theta + C = \sin^{-1}\left(\frac{x}{a}\right) + C$$

This suggests the following,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

∎

Remark 7.4.2.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Chap. 7 / Sec. 7.4 / Subsec. 7.4.2 : Integrals Involving $\sqrt{a^2 + x^2}$

Suppose, we're given an integral whose integrand involves the term $\sqrt{a^2 + x^2}$, where a is a positive constant. If we substitute $x = a\tan(\theta)$, then

$$
\begin{aligned}
\sqrt{a^2 + x^2} &= \sqrt{a^2 + (a\tan(\theta))^2} \\
&= \sqrt{a^2 + a^2\tan^2(\theta)} \\
&= \sqrt{a^2(1 + \tan^2(\theta))}
\end{aligned}
$$

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It is known from the Pythagorean identity that $\sec^2(x) = 1+\tan^2(x)$. Therefore,

$$\begin{aligned} &= \sqrt{a^2 \underbrace{\left(1+\tan^2(\theta)\right)}_{\sec^2(\theta)}} \\ &= \sqrt{a^2 \cdot \sec^2(\theta)} \\ &= |a \cdot \sec(\theta)| \\ &= a\sec(\theta) \end{aligned} \tag{2}$$

This implies that the substitution $x = a\tan(\theta)$ transform the difficult $\sqrt{a^2+x^2}$ into an easier product $a\sec(\theta)$. That is,

$$\sqrt{a^2+x^2} = a\sec(\theta)$$

if we let $x = a\tan(\theta)$.

The substitution $x = a\tan(\theta)$ for $\sqrt{a^2+x^2}$ works because, if we consider a right-angled triangle whose opposite side is x and adjacent side is a, then the hypotenuse is given by the Pythagorean theorem, as follows.

$$\begin{aligned} \text{opposite side}^2 + \text{adjacent side}^2 &= \text{hypotenuse}^2 \\ x^2 + a^2 &= \text{hypotenuse}^2 \\ \therefore \text{hypotenuse} &= \sqrt{x^2+a^2} \end{aligned}$$

This is shown below.

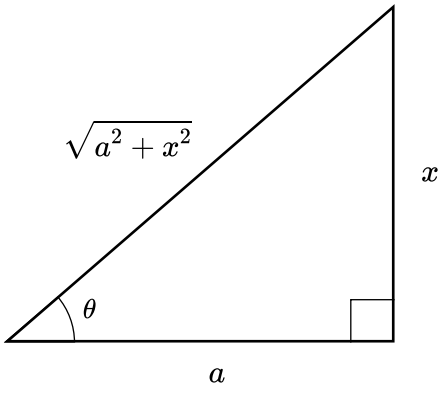

Figure 7.2

This implies that

$$\tan(\theta) = \frac{x}{a}$$

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$$\therefore x = a\tan(\theta)$$

A few examples are provided below.

► **Example 7.4.5.** Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$

Solution. Notice that the integral above involves the term $\sqrt{x^2+a^2}$, where $a=3$. Therefore, to solve the integral, we let $x = 3\tan(\theta)$, then $dx = 3\sec^2(\theta)d\theta$. Therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\sec^2(\theta)d\theta}{\sqrt{(3\tan(\theta))^2+9}} \\
&= \int \frac{3\sec^2(\theta)d\theta}{\sqrt{9\tan^2(\theta)+9}} \\
&= \int \frac{3\sec^2(\theta)d\theta}{\sqrt{9\left(\tan^2(\theta)+1\right)}} \\
&= \int \frac{3\sec^2(\theta)d\theta}{\sqrt{9}\cdot\sqrt{\sec^2(\theta)}} \\
&= \int \frac{\cancel{3}\sec^2(\theta)d\theta}{\cancel{3}\sec(\theta)} \\
&= \int \sec(\theta)d\theta
\end{aligned}$$

Since $\displaystyle\int \sec(\theta)d\theta = \ln\left|\sec(\theta)+\tan(\theta)\right| + C$ and if $x = 3\tan(\theta)$, then $\theta = \tan^{-1}\left(\frac{x}{3}\right)$, therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \ln\left|\sec(\theta)+\tan(\theta)\right| + C \\
&= \ln\left|\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x}{3}\right)\right)\right| + C
\end{aligned}$$

Since $\tan(\tan^{-1}(x)) = x$ and $\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$, therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \ln\left|\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x}{3}\right)\right)\right| + C \\
&= \ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2} + \frac{x}{3}\right| + C \\
&= \ln\left|\sqrt{1+\frac{x^2}{9}} + \frac{x}{3}\right| + C
\end{aligned}$$

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$$= \ln\left|\sqrt{\frac{9+x^2}{9}} + \frac{x}{3}\right| + C$$
$$= \ln\left|\frac{\sqrt{x^2+9}}{\sqrt{9}} + \frac{x}{3}\right| + C$$
$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right| + C$$

Therefore, $\displaystyle\int \frac{dx}{\sqrt{x^2+9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right| + C$ ∎

We now generalize the integrals above into formulas.

► **Example 7.4.6.** Evaluate $\displaystyle\int \frac{dx}{a^2+x^2}$

Solution. To evaluate the integral, we let $x = a\tan(\theta)$. Then, the differential of x becomes $dx = a\sec^2(\theta)d\theta$. Therefore,

$$\begin{aligned}\int \frac{dx}{a^2+x^2} &= \int \frac{dx}{a^2+(a\tan(x))^2} \\ &= \int \frac{a\sec^2(\theta)\,d\theta}{a^2+a^2\tan^2(\theta)} \\ &= \int \frac{a\sec^2(\theta)\,d\theta}{a^2\sec^2(\theta)} \\ &= \frac{1}{a}\int d\theta \\ &= \frac{\theta}{a} + C \end{aligned} \tag{E1}$$

Now, since $x = a\tan(\theta)$, therefore,

$$\begin{aligned} x &= a\tan(\theta) \\ \frac{x}{a} &= \tan(\theta) \\ \therefore \theta &= \tan^{-1}(\frac{x}{a}) \end{aligned}$$

Substituting $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ into equation (E1) yields the following,

$$\frac{\theta}{a} + C = \frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} + C$$

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This suggests the following,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

∎

Remark 7.4.3.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

► **Example 7.4.7.** Evaluate $\int \sqrt{a^2+x^2}dx$

Solution. To evaluate the integral, we let $x = a\tan(\theta)$. Then, the differential dx yields $dx = a\sec^2(\theta)d\theta$. Therefore,

$$\begin{aligned}
\int \sqrt{a^2+x^2}dx &= \int \sqrt{a^2+(a\tan(\theta))^2}a\sec^2(\theta)\,d\theta \\
&= \int \sqrt{a^2+a^2\tan^2(\theta)}a\sec^2(\theta)\,d\theta \\
&= \int \sqrt{a^2(1+\tan^2(\theta))}a\sec^2(\theta)\,d\theta \\
&= \int \sqrt{a^2\sec^2(\theta)}a\sec^2(\theta)\,d\theta \\
&= \int (a\sec(\theta))\left(a\sec^2(\theta)\right)d\theta \\
&= \int a^2\sec^3(\theta)\,d\theta \\
&= a^2\int \sec^3(\theta)\,d\theta \qquad \text{(E1)}
\end{aligned}$$

Now, to evaluate $\int \sec^3(\theta)d\theta$ in the integral above, we apply the reduction formula, stating that,

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

for $n = 3$. That is,

$$\begin{aligned}
\int \sec^3(x)\,dx &= \frac{\sec(x)\tan(x)}{2} + \frac{1}{2}\int \sec(x)\,dx \\
&= \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\tan(x)+\sec(x)|}{2} + C
\end{aligned}$$

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Now, substituting $\frac{\sec(\theta)\tan(\theta)}{2} + \frac{\ln|\tan(\theta)+\sec(\theta)|}{2} + C$ for $\int \sec^3(\theta)\,dx$ in the equation (E1), we get,

$$\begin{aligned} a^2 \int \sec^3(\theta)\,d\theta &= a^2\left(\frac{\sec(\theta)\tan(\theta)}{2} + \frac{\ln|\tan(\theta)+\sec(\theta)|}{2} + C\right) \\ &= \frac{a^2}{2}\left(\sec(\theta)\tan(\theta) + \ln|\tan(\theta)+\sec(\theta)|\right) + C \end{aligned}$$

This implies,

$$\int \sqrt{a^2+x^2}dx = \frac{a^2}{2}\left(\sec(\theta)\tan(\theta) + \ln|\tan(\theta)+\sec(\theta)|\right) + C \tag{E2}$$

Now, from the Pythagorean theorem, it is known that,

$$\begin{aligned} \sec^2(x) - \tan^2(x) &= 1 \\ \sec^2(x) &= 1 + \tan^2(x) \\ \therefore \sec(x) &= \sqrt{1+\tan^2(x)} \end{aligned} \tag{E3}$$

Substituting equation (E3) into (E2) yields the following.

$$\begin{aligned} &= \frac{a^2}{2}\left(\sec(\theta)\tan(\theta) + \ln|\tan(\theta)+\sec(\theta)|\right) + C \\ &= \frac{a^2}{2}\left(\sqrt{1+\tan^2(\theta)}\tan(\theta) + \ln\left|\tan(\theta) + \sqrt{1+\tan^2(\theta)}\right|\right) + C \end{aligned} \tag{E4}$$

Now, to express the integral in terms of x, notice that $x = a\tan(\theta)$, therefore,

$$\begin{aligned} x &= a\tan(\theta) \\ \frac{x}{a} &= \tan(\theta) \\ \therefore \theta &= \tan^{-1}\left(\frac{x}{a}\right) \end{aligned}$$

Substituting $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ into equation (E4), we get,

$$\begin{aligned} =&\frac{a^2}{2}\left(\sqrt{1+\tan^2(\theta)}\tan(\theta) + \ln\left|\tan(\theta) + \sqrt{1+\tan^2(\theta)}\right|\right) + C \\ =&\frac{a^2}{2}\left(\sqrt{1+\tan^2\left(\tan^{-1}\left(\frac{x}{a}\right)\right)}\tan\left(\tan^{-1}\left(\frac{x}{a}\right)\right)\right. \\ &\left. + \ln\left|\tan\left(\tan^{-1}\left(\frac{x}{a}\right)\right) + \sqrt{1+\tan^2\left(\tan^{-1}\left(\frac{x}{a}\right)\right)}\right|\right) + C \\ =&\frac{a^2}{2}\left(\sqrt{1+\left(\frac{x}{a}\right)^2}\cdot\frac{x}{a} + \ln\left|\frac{x}{a} + \sqrt{1+\left(\frac{x}{a}\right)^2}\right|\right) + C \end{aligned}$$

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$$
\begin{aligned}
&= \frac{a^2}{2}\left(\sqrt{1+\frac{x^2}{a^2}}\cdot\frac{x}{a}+\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|\right)+C\\
&= \frac{1}{2}\left(a^2\sqrt{\frac{a^2+x^2}{a^2}}\cdot\frac{x}{a}+a^2\ln\left|\frac{x}{a}+\sqrt{\frac{a^2+x^2}{a^2}}\right|\right)+C\\
&= \frac{1}{2}\left(a^2\sqrt{\frac{1}{a^2}}\cdot\sqrt{a^2+x^2}\cdot\frac{x}{a}+a^2\ln\left|\frac{x}{a}+\sqrt{\frac{1}{a^2}}\cdot\sqrt{a^2+x^2}\right|\right)+C\\
&= \frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left|\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}\right|\right)+C\\
&= \frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right)+C
\end{aligned}
$$

This suggests the following,

$$\int\sqrt{a^2+x^2}dx=\frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right)+C$$

∎

Remark 7.4.4.

$$\int\sqrt{a^2+x^2}dx=\frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right)+C$$

Chap. 7 / Sec. 7.4 / Subsec. 7.4.3 : Integrals Involving $\sqrt{x^2-a^2}$

Suppose, we're given an integral whose integrand involves the term $\sqrt{x^2-a^2}$, where a is a positive constant. If we substitute $x=a\sec(\theta)$, then

$$
\begin{aligned}
\sqrt{x^2-a^2} &= \sqrt{\left(a\sec\left(\theta\right)\right)^2-a^2}\\
&= \sqrt{a^2\sec^2\left(\theta\right)-a^2}\\
&= \sqrt{a^2\left(\sec^2\left(\theta\right)-1\right)}
\end{aligned}
$$

It is known from the Pythagorean identity that $\sec^2(x)=1+\tan^2(x)$ or, $\sec^2(x)-1=\tan^2(x)$. Therefore,

$$= \sqrt{a^2\underbrace{\left(\sec^2\left(\theta\right)-1\right)}_{\tan^2(\theta)}}$$

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$$
\begin{aligned}
&= \sqrt{a^2 \cdot \tan^2(\theta)} \\
&= |a \cdot \tan(\theta)| \\
&= a \tan(\theta)
\end{aligned}
$$

This implies that the substitution $x = a\sec(\theta)$ transforms the difficult $\sqrt{x^2 - a^2}$ into an easier product $a\tan(\theta)$. That is,

$$\sqrt{x^2 - a^2} = a\tan(\theta)$$

if we let $x = a\tan(\theta)$.

The reason why the substitution $x = a\sec(\theta)$ works is that if we consider a right-angled triangle whose hypotenuse is x and the adjacent side is a, then the opposite side is given by the Pythagorean theorem, stating that,

$$
\begin{aligned}
\text{opposite side}^2 + \text{adjacent side}^2 &= \text{hypotenuse}^2 \\
\text{opposite side}^2 + a^2 &= x^2 \\
\text{opposite side} &= \sqrt{x^2 - a^2}
\end{aligned}
$$

This is shown below.

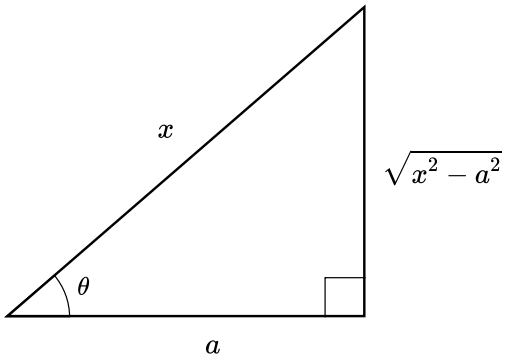

Figure 7.3

This implies that

$$
\begin{aligned}
\sec(\theta) &= \frac{x}{a} \\
\therefore x &= a\sec(\theta)
\end{aligned}
$$

A few examples are provided below.

▶ **Example 7.4.8.** Evaluate $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 16}}$

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Solution. Notice that the integral above involves the term $\sqrt{x^2-a^2}$, where $a=4$. Therefore, to solve the integral, we let $x=4\sec(\theta)$, then $dx=4\sec(\theta)\tan(\theta)d\theta$. Thus,

$$\begin{aligned}
\int \frac{xdx}{\sqrt{x^2-16}} &= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{\sqrt{(4\sec(\theta))^2-16}} \\
&= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{\sqrt{16\sec^2(\theta)-16}} \\
&= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{\sqrt{16\left(\sec^2(\theta)-1\right)}} \\
&= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{\sqrt{16}\cdot\sqrt{\sec^2(\theta)-1}} \\
&= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{4\cdot\sqrt{\tan^2(\theta)}} \\
&= \int \frac{4\sec(\theta)\cdot 4\sec(\theta)\tan(\theta)d\theta}{4\tan(\theta)} \\
&= \int \frac{\cancel{4}\sec(\theta)\cdot 4\sec(\theta)\cancel{\tan(\theta)}d\theta}{\cancel{4}\cancel{\tan(\theta)}} \\
&= \int 4\sec^2(\theta)d\theta \\
&= 4\int \sec^2(\theta)d\theta \\
&= 4\tan^2(\theta)+C
\end{aligned}$$

Since $x=4\sec(\theta)$, therefore $\theta=\sec^{-1}\left(\frac{x}{4}\right)$. Thus,

$$\begin{aligned}
\int \frac{xdx}{\sqrt{x^2-16}} &= 4\tan^2(\theta)+C \\
&= 4\tan^2\left(\sec^{-1}\left(\frac{x}{4}\right)\right)+C
\end{aligned}$$

Since $\tan(\sec^{-1})=\sqrt{x^1-1}$, therefore,

$$\begin{aligned}
\int \frac{xdx}{\sqrt{x^2-16}} &= 4\tan^2\left(\sec^{-1}\left(\frac{x}{4}\right)\right)+C \\
&= 4\cdot\sqrt{\left(\frac{x}{4}\right)^2-1}+C \\
&= 4\cdot\sqrt{\frac{x^2-16}{16}}+C \\
&= \sqrt{x^2-16}+C
\end{aligned}$$

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Therefore, $\displaystyle\int \frac{xdx}{\sqrt{x^2-16}} = \sqrt{x^2-16}+C$ ∎

We now generalize the integrals above into formulas.

► **Example 7.4.9.** Evaluate $\displaystyle\int \sqrt{x^2-a^2}dx$

Solution. To evaluate the integral, we let $x = a\sec(\theta)$. Then, the differential dx yields $dx = a\sec(\theta)\tan(\theta)d\theta$. Therefore,

$$\begin{aligned}
\int \sqrt{x^2-a^2}dx &= \int \sqrt{(a\sec(\theta))^2-a^2}a\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \sqrt{(a\sec(\theta))^2-a^2}a\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \sqrt{a^2\sec^2(\theta)-a^2}a\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \sqrt{a^2(\sec^2(\theta)-1)}a\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \sqrt{a^2}\sqrt{\tan^2(\theta)}a\sec(\theta)\tan(\theta)\,d\theta \\
&= a^2\int \sec(\theta)\tan^2(\theta)\,d\theta \\
&= a^2\int \sec(\theta)\left(\sec^2(\theta)-1\right)d\theta \\
&= a^2\int \left(\sec^3(\theta)-\sec(\theta)\right)d\theta \\
&= a^2\int \sec^3(\theta)\,d\theta - a^2\int \sec(\theta)\,d\theta \qquad \text{(E1)}
\end{aligned}$$

Now, to evaluate $\displaystyle\int \sec^3(\theta)d\theta$ in the integral above, we apply the reduction formula:

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

for $n = 3$. That is,

$$\begin{aligned}
\int \sec^3(x)\,dx &= \frac{\sec(x)\tan(x)}{2} + \frac{1}{2}\int \sec(x)\,dx \\
&= \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\tan(x)+\sec(x)|}{2} + C
\end{aligned}$$

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Now, substituting $\frac{\sec(\theta)\tan(\theta)}{2}+\frac{\ln|\tan(\theta)+\sec(\theta)|}{2}+C$ for $\int \sec^3(\theta)\,dx$ and $\ln|\tan(\theta)+\sec(\theta)|$ for $\int \sec(\theta)d\theta$ in the equation (E1) yields the following.

$$
\begin{aligned}
&= a^2\int \sec^3(\theta)\,d\theta - a^2\int \sec(\theta)\,d\theta \\
&= a^2\left(\frac{\sec(\theta)\tan(\theta)}{2}+\frac{\ln|\tan(\theta)+\sec(\theta)|}{2}\right) - a^2\ln|\tan(\theta)+\sec(\theta)| + C \\
&= \frac{a^2\sec(\theta)\tan(\theta)}{2}+\frac{a^2\ln|\tan(\theta)+\sec(\theta)|}{2} - \frac{2a^2\ln|\tan(\theta)+\sec(\theta)|}{2} + C \\
&= \frac{a^2\sec(\theta)\tan(\theta)+a^2\ln|\tan(\theta)+\sec(\theta)| - 2a^2\ln|\tan(\theta)+\sec(\theta)|}{2} + C \\
&= \frac{a^2\sec(\theta)\tan(\theta)-a^2\ln|\tan(\theta)+\sec(\theta)|}{2} + C \\
&= \frac{a^2\left(\sec(\theta)\tan(\theta)-\ln|\tan(\theta)+\sec(\theta)|\right)}{2} + C \\
&= \frac{a^2}{2}\left(\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\right) + C
\end{aligned}
$$

This implies,

$$\int \sqrt{x^2-a^2}dx = \frac{a^2}{2}\left(\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\right)+C \tag{E2}$$

Now, from the Pythagorean theorem, it is known that,

$$
\begin{aligned}
\sec^2(x)-\tan^2(x) &= 1 \\
\tan^2(x) &= \sec^2(x)-1 \\
\tan(x) &= \sqrt{\sec^2(x)-1}
\end{aligned} \tag{E3}
$$

Substituting equation (E3) into (E2) yields the following.

$$
\begin{aligned}
&= \frac{a^2}{2}\left(\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\right)+C \\
&= \frac{a^2}{2}\left(\sec(\theta)\sqrt{\sec^2(\theta)-1}-\ln|\sec(\theta)+\sqrt{\sec^2(\theta)-1}|\right)+C
\end{aligned} \tag{E4}
$$

Now, to express the integral in terms of x, notice that $x = a\sec(\theta)$, therefore,

$$
\begin{aligned}
x &= a\sec(\theta) \\
\frac{x}{a} &= \sec(\theta) \\
\therefore \theta &= \sec^{-1}\left(\frac{x}{a}\right)
\end{aligned}
$$

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Substituting $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ into equation (E4), we obtain the following,

$$
\begin{aligned}
=&\frac{a^2}{2}\left(\sec(\theta)\sqrt{\sec^2(\theta)-1} - \ln\left|\sec(\theta)+\sqrt{\sec^2(\theta)-1}\right|\right)+C\\
=&\frac{a^2}{2}\left(\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right)\sqrt{\sec^2\left(\sec^{-1}\left(\frac{x}{a}\right)\right)-1}\right.\\
&\left.-\ln\left|\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right)+\sqrt{\sec^2\left(\sec^{-1}\left(\frac{x}{a}\right)\right)-1}\right|\right)+C\\
=&\frac{a^2}{2}\left(\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}-\ln\left|\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right|\right)+C\\
=&\frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1}-\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|\right)+C\\
=&\frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{x^2-a^2}{a^2}}-\ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|\right)+C\\
=&\frac{1}{2}\left(a^2\cdot\frac{x}{a}\sqrt{\frac{x^2-a^2}{a^2}}-a^2\cdot\ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|\right)+C\\
=&\frac{1}{2}\left(a^2\cdot\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a}-a^2\cdot\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right)+C\\
=&\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|\right)+C
\end{aligned}
$$

This suggests the following.

$$\int\sqrt{x^2-a^2}dx=\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|\right)+C$$

∎

Remark 7.4.5.

$$\int\sqrt{x^2-a^2}dx=\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|\right)+C$$

The End of Section 7.4
Trigonometric Substitution

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§ 7.5 Partial Fractions

Partial fraction decomposition is a powerful technique used in calculus to integrate rational functions. For example, consider the following function,

$$f(x) = \frac{P(x)}{Q(x)}$$

where, $P(x)$ and $Q(x) \neq 0$ are both polynomials. In that case, $f(x)$ is said to be a rational function and is expressed as the ratio of polynomial functions. A polynomial is a function of the form,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where, n represents a non-negative integer and $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constants, where $a_n \neq 0$.

Integrating functions of this nature isn't always straightforward. It is beneficial to decompose them into simpler functions, which can be easily integrated. For example, suppose we have to integrate the following integral,

$$\int \frac{6x}{x^2 + x - 2} dx$$

Other methods to integrate the integral above might exist, but no method is as simple as if we realize that $\dfrac{6x}{x^2 + x - 2}$ can be written as $\dfrac{2}{x-1} + \dfrac{4}{x+2}$ for the reason that,

$$\begin{aligned}
\frac{2}{x-1} + \frac{4}{x+2} &= \frac{2(x+2) + 4(x-1)}{(x-1)(x+2)} \\
&= \frac{2x + \cancel{4} + 4x - \cancel{4}}{(x-1)(x+2)} \\
&= \frac{2x + 4x}{(x-1)(x+2)} \\
&= \frac{6x}{(x-1)(x+2)} \\
&= \frac{6x}{x^2 + 2x - x - 2} \\
&= \frac{6x}{x^2 + x - 2}
\end{aligned}$$

This suggests that,

$$\int \frac{2x}{x^2 + x - 2} dx = \int \left(\frac{2}{x-1} + \frac{4}{x+2} \right) dx$$

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Now, notice that the integral on the right side of the equation is straightforward to solve, whereas the integral on the left side is not. That is,

$$\begin{aligned}\int \left(\frac{2}{x-1}+\frac{4}{x+2}\right)dx &= \int \frac{2}{x-1}dx + \int \frac{4}{x+2}dx \\ &= 2\int \frac{1}{x-1}dx + 4\int \frac{1}{x+2}dx \\ &= 2\ln|x-1| + 4|x+2| + C \\ \therefore \int \frac{2x}{x^2+x-2}dx &= 2\ln|x-1| + 4|x+2| + C\end{aligned}$$

While the method is efficient, its practicality relies on the ability to factor $\frac{6x}{x^2+x-2}$ into fractions $\frac{2}{x-1}$ and $\frac{4}{x+2}$, referred to as *partial fractions*. For example, if $\frac{2}{x-1}$ and $\frac{4}{x+2}$ are given, then it is of no effort that we may determine $\frac{6x}{x^2+x-2}$, however, the converse is not true. Therefore, in this section, various techniques are discussed regarding that issue.

However, that comes with a few considerations. First of all, we can only decompose a rational function $\frac{P(x)}{Q(x)}$ if the degree of $P(x)$ is less than that of $Q(x)$, symbolically, $\deg\Big(P(x)\Big) < \deg\Big(Q(x)\Big)$. Such a rational function is called a *proper* rational function. In contrast, if we ever face an *improper* rational function, where the degree of $P(x)$ is greater than $Q(x)$ or equal to that, that is, $\deg\Big(P(x)\Big) \geq \deg\Big(Q(x)\Big)$, then we'll first need to perform long polynomial division to express $\frac{P(x)}{Q(x)}$ as $S(x) + \frac{R(x)}{Q(x)}$, where $R(x)$ is the remainder and $\deg\Big(R(x)\Big) < \deg\Big(Q(x)\Big)$.

So, in general, suppose $P(x)$ and $Q(x)$ are both polynomial functions, where $\deg\Big(P(x)\Big) < \deg\Big(Q(x)\Big)$, also suppose that $Q(x)$ can be factored as a product of linear factors (of the form $ax+b$) or/and irreducible quadratic factors (of the form ax^2+bx+c, where $b^2-4ac<0$); then, $\frac{P(x)}{Q(x)}$ can be expressed as a sum of partial fractions.

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Chap. 7 / Sec. 7.5 / Subsec. 7.5.1 : The denominator $Q(x)$ is a product of linear factors

Remark 7.5.1. If $Q(x)$ in the rational function $\dfrac{P(x)}{Q(x)}$ can be factored into a product of linear factors, where each linear factor is distinct, that is, factors of the form,

$$(a_1x + b_1)\,(a_2x + b_2)\cdots(a_nx + b_n)$$

then, there exist constant $A_1, A_2, \cdots, A_n$ such that the rational function $\dfrac{P(x)}{Q(x)}$ has a partial fraction decomposition of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x + b_1)\,(a_2x + b_2)\cdots(a_nx + b_n)}\\ &= \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}\end{aligned}$$

► **Example 7.5.1.** Evaluate $\displaystyle\int \frac{x^2 + 2x + 1}{x^3 + 2x^2 + x^2 + 2x}dx$

Solution. In order to integrate the integral above, notice first that $x^3 + 2x^2 + x^2 + 2x$ can be factored as $x(x + 1)(x + 2)$. As a result,

$$\begin{aligned}\frac{x^2 + 2x + 1}{x^3 + 2x^2 + x^2 + 2x} &= \frac{x^2 + 2x + 1}{x(x + 1)(x + 2)}\\ &= \frac{A}{x} + \frac{B}{(x + 1)} + \frac{C}{(x + 2)}\end{aligned}$$

Now, the goal is to determine the values of A, B, and C. For that multiply $x(x + 1)(x + 2)$ in the both sides of the equation below,

$$\begin{aligned}\frac{x^2 + 2x + 1}{x(x + 1)(x + 2)} &= \frac{A}{x} + \frac{B}{(x + 1)} + \frac{C}{(x + 2)}\\ x^2 + 2x + 1 &= \frac{A}{\cancel{x}}\cdot\cancel{x}(x + 1)(x + 2) + \frac{B}{\cancel{(x + 1)}}\cdot x\cancel{(x + 1)}(x + 2) + \frac{C}{\cancel{(x + 2)}}\cdot x(x + 1)\cancel{(x + 2)}\\ x^2 + 2x + 1 &= A(x + 1)(x + 2) + Bx(x + 2) + Cx(x + 1)\\ &= A(x^2 + 2x + x + 2) + B(x^2 + 2x) + C(x^2 + x)\\ &= Ax^2 + 2Ax + Ax + 2A + Bx^2 + 2Bx + Cx^2 + Cx\\ &= (Ax^2 + Bx^2 + Cx^2) + (2Ax + Ax + 2Bx + Cx) + 2A\\ x^2 + 2x + 1 &= (A + B + C)x^2 + (3A + 2B + C)x + 2A\end{aligned}$$

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Since $(A+B+C)x^2+(3A+2B+C)x+2A$ is identical to x^2+2x+1, therefore, their coefficients must also be equal. Thus, we write the following,

$$A+B+C=1 \tag{1}$$
$$3A+2B+C=2 \tag{2}$$
$$2A=1 \tag{3}$$

Since $2A=1$, therefore, $A=\frac{1}{2}$. Substituting the value of A into equation (1) yields the following,

$$\frac{1}{2}+B+C=1$$
$$B=1-C-\frac{1}{2} \tag{4}$$

Substituting the value of A and B into equation (2) yields the following,

$$3\left(\frac{1}{2}\right)+2\left(1-C-\frac{1}{2}\right)+C=2$$
$$C=2-2\left(1-C-\frac{1}{2}\right)-3\left(\frac{1}{2}\right)$$
$$=2-2+2C+1-\frac{3}{2}$$
$$=2C+1-\frac{3}{2}$$
$$=2C-\frac{1}{2}$$
$$C-2C=-\frac{1}{2}$$
$$-C=-\frac{1}{2}$$
$$\therefore C=\frac{1}{2}$$

Substituting the value of A and C into equation (4) yields the following,

$$B=1-C-\frac{1}{2}$$
$$=0$$

Therefore, we have found the value of A, B, and C to be $\frac{1}{2}$, 0, and $\frac{1}{2}$, respectively. Thus, we write,

$$\frac{x^2+2x+1}{x(x+1)(x+2)}=\frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x+2)}$$

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$$= \frac{\frac{1}{2}}{x} + \frac{0}{(x+1)} + \frac{\frac{1}{2}}{(x+2)}$$
$$= \frac{1}{2x} + \frac{1}{2(x+2)}$$

As a result,

$$\begin{aligned}\int \frac{x^2+2x+1}{x^3+2x^2+x^2+2x}dx &= \int \left(\frac{1}{2x} + \frac{1}{2(x+2)}\right)dx \\ &= \int \frac{1}{2x}dx + \int \frac{1}{2(x+2)}dx \\ &= \frac{1}{2}\cdot\int \frac{1}{x}dx + \frac{1}{2}\cdot\int \frac{1}{x+2}dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + C\end{aligned}$$

Therefore, $\int \frac{x^2+2x+1}{x^3+2x^2+x^2+2x}dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + C$ ∎

Chap. 7 / Sec. 7.5 / Subsec. 7.5.2 : The denominator $Q(x)$ contains repeated product of linear factors

Remark 7.5.2. If $Q(x)$ in the rational function $\frac{P(x)}{Q(x)}$ has at least one repeated liner factor, that is, factor of the form,

$$(ax+b)^n \qquad \text{where } n \geq 2$$

then, there exist constants $A_1, A_2, \cdots, A_n$ such that the rational function $\frac{P(x)}{Q(x)}$ has a partial fraction decomposition of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(ax+b)^n} \\ &= \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}\end{aligned}$$

► **Example 7.5.2.** Evaluate $\int \frac{x^2+x+1}{(x+1)^2(x-1)}dx$

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Solution. In order to integrate the integral above, we express the given rational function as a sum of partial fractions as follows.

$$\frac{x^2+x+1}{(x+1)^2(x-1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Now, multiply both sides by the denominator $(x+1)^2(x-1)$ to eliminate the denominators.

$$\begin{aligned} x^2+x+1 &= A(x+1)^2 + B(x+1)(x-1) + C(x-1) \\ &= A(x^2+2x+1) + B(x^2-1) + C(x-1) \\ &= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C \\ &= Ax^2 + Bx^2 + 2Ax + Cx + A - B - C \\ \therefore x^2+x+1 &= (A+B)x^2 + (2A+C)x + (A-B-C) \end{aligned}$$

Now, we compare the coefficients of x^2, x, and the constant terms on both sides of the equations, as follows.

$$A + B = 1 \tag{1}$$
$$2A + C = 1 \tag{2}$$
$$A - B - C = 1 \tag{3}$$

Now, to determine the values of A, B, and C, notice that since $A + B = 1$, therefore, $B = 1 - A$. Substituting the value of B into equation (3) yields the following.

$$\begin{aligned} A - B - C &= 1 \\ A - (1-A) - C &= 1 \\ A - 1 + A - C &= 1 \\ 2A - C &= 2 \end{aligned} \tag{4}$$

Adding equations (2) and (4) together, we get,

$$\begin{aligned} 4A &= 3 \\ \boxed{A = \frac{3}{4}} \end{aligned}$$

Substituting $A = \dfrac{3}{4}$ into equation (2), we get,

$$2\left(\frac{3}{4}\right) + C = 1$$

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$$\begin{aligned}\frac{3}{2}+C&=1\\ C&=1-\frac{3}{2}\\ &=-\frac{1}{2}\end{aligned}$$

This implies that $\boxed{C=-\frac{1}{2}}$. Substituting the value of A into equation (1) yields the following,

$$\begin{aligned}A+B&=1\\ B&=1-\frac{3}{4}\\ &=\frac{1}{4}\end{aligned}$$

This implies that $\boxed{B=\frac{1}{4}}$. Thus, we have the partial fraction decomposition as follows,

$$\frac{x^2+x+1}{(x+1)^2(x-1)}=\frac{\frac{3}{4}}{(x-1)}+\frac{\frac{1}{4}}{(x+1)}-\frac{\frac{1}{2}}{(x+1)^2}$$

This implies that

$$\frac{x^2+x+1}{(x+1)^2(x-1)}=\frac{\frac{3}{4}}{(x-1)}+\frac{\frac{1}{4}}{(x+1)}-\frac{\frac{1}{2}}{(x+1)^2}$$

$$\begin{aligned}\int\frac{x^2+x+1}{(x+1)^2(x-1)}dx&=\int\left(\frac{\frac{3}{4}}{(x-1)}+\frac{\frac{1}{4}}{(x+1)}-\frac{\frac{1}{2}}{(x+1)^2}\right)dx\\ &=\int\frac{\frac{3}{4}}{(x-1)}dx+\int\frac{\frac{1}{4}}{(x+1)}dx-\int\frac{\frac{1}{2}}{(x+1)^2}dx\\ &=\frac{3}{4}\cdot\int\frac{1}{(x-1)}dx+\frac{1}{4}\cdot\int\frac{1}{(x+1)}dx-\frac{1}{2}\cdot\int\frac{1}{(x+1)^2}dx\\ &=\frac{3}{4}\ln|x-1|+C_1+\frac{1}{4}\ln|x+1|+C_2-\frac{1}{2}\cdot\int\frac{1}{(x+1)^2}dx\end{aligned}\tag{5}$$

Now, to solve the integral $\int\frac{1}{(x+1)^2}dx$, we use the u-substitution rule. There-

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fore, let $u = x + 1$, then $du = dx$, thus, we write the following,

$$\begin{aligned}\int \frac{1}{(x+1)^2} dx &= \int \frac{1}{u^2} du \\ &= \int u^{-2} du \\ &= \frac{u^{-2+1}}{-2+1} + C_3 \\ &= \frac{u^{-1}}{-1} + C_3 \\ &= -\frac{1}{u} + C_3 \\ &= -\frac{1}{x+1} + C_3 \end{aligned} \tag{6}$$

Now, substituting the result in equation 6 into equation 5, we get,

$$\begin{aligned}\int \frac{x^2+x+1}{(x+1)^2(x-1)} dx &= \frac{3}{4}\ln|x-1| + C_1 + \frac{1}{4}\ln|x+1| + C_2 - \frac{1}{2}\cdot\int \frac{1}{(x+1)^2} dx \\ &= \frac{3}{4}\ln|x-1| + C_1 + \frac{1}{4}\ln|x+1| + C_2 - \frac{1}{2}\cdot\left(-\frac{1}{x+1} + C_3\right) \\ &= \frac{3}{4}\ln|x-1| + C_1 + \frac{1}{4}\ln|x+1| + C_2 + \frac{1}{2(x+1)} + C_3 \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + C\end{aligned}$$

where, $C = C_1 + C_2 + C_3$. Therefore, $\int \frac{x^2+x+1}{(x+1)^2(x-1)} dx = \frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$.

∎

Chap. 7 / Sec. 7.5 / Subsec. 7.5.3 : The denominator $Q(x)$ contains irreducible quadratic factors

Remark 7.5.3. If $Q(x)$ in the rational function $\frac{P(x)}{Q(x)}$ has at least one irreducible quadratic factor, that s, factor of the form,

$$ax^2 + bx + c \qquad \text{where, } b^2 - 4ac < 0$$

then, there exist constants A and B such that the rational function $\frac{P(x)}{Q(x)}$

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has a partial fraction decomposition of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{ax^2+bx+c} \\ &= \frac{Ax+B}{ax^2+bx+c}\end{aligned}$$

► **Example 7.5.3.** Evaluate $\displaystyle\int \frac{x^2+x+5}{x^3+5x}dx$

Solution. Notice that x^3+5x can be factored as $x(x^2+5)$. Since x^2+5 cannot be factored any further as $b^2-4ac<0$, therefore we write the integrand as,

$$\begin{aligned}\frac{x^2+x+5}{x^3+5x} &= \frac{x^2+x+5}{x(x^2+5)} \\ &= \frac{A}{x}+\frac{Bx+C}{x^2+5}\end{aligned}$$

Now, we multiply both sides by the denominator x^3+5x to eliminate the denominators. That is,

$$\begin{aligned}x^2+x+5 &= A(x^2+5)+(Bx+C)x \\ &= Ax^2+5A+Bx^2+Cx \\ &= Ax^2+Bx^2+Cx+5A \\ &= (A+B)x^2+Cx+5A\end{aligned}$$

Now, we compare the coefficients of x^2, x, and the constant term both sides of the equation,

$$A+B=1 \tag{1}$$
$$C=1 \tag{2}$$
$$5A=5 \tag{3}$$

It is given that $\boxed{C=1}$. Since $5A=5$, therefore $\boxed{A=1}$. Substituting the value of A into equation (1), we get the value of B,

$$\begin{aligned}A+B &= 1 \\ B &= 0\end{aligned}$$

Therefore, we have found the value of A, B, and C to be 1, 0, and 1, respectively. Thus, we write,

$$\frac{x^2+x+5}{x^3+5x} = \frac{A}{x}+\frac{Bx+C}{x^2+5}$$

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$$\begin{aligned}&= \frac{1}{x} + \frac{0 \cdot x + 1}{x^2 + 5} \\ &= \frac{1}{x} + \frac{1}{x^2 + 5}\end{aligned}$$

As a result,

$$\begin{aligned}\int \frac{x^2 + x + 5}{x^3 + 5x} dx &= \int \left(\frac{1}{x} + \frac{1}{x^2 + 5} \right) dx \\ &= \int \frac{1}{x} dx + \int \frac{1}{x^2 + 5} dx\end{aligned}$$

Now, to solve the integral $\int \frac{1}{x^2 + 5} dx$, we let $u = \frac{x}{\sqrt{5}}$, then $x = \sqrt{5}u$ and $du = \frac{1}{\sqrt{5}} dx$. Thus, we get,

$$\begin{aligned}\int \frac{1}{x^2 + 5} dx &= \int \frac{\sqrt{5}}{\left(\sqrt{5}u\right)^2 + 5} du \\ &= \int \frac{\sqrt{5}}{5\left(u^2 + 1\right)} du \\ &= \frac{\sqrt{5}}{5} \cdot \int \frac{1}{u^2 + 1} du\end{aligned}$$

Now, to solve the integral $\int \frac{1}{u^2 + 1} du$, we let $u = \tan(z)$, then $du = \sec^2(z) dz$. Thus,

$$\begin{aligned}\int \frac{1}{u^2 + 1} du &= \int \frac{\sec^2(z)}{\tan^2(z) + 1} dz \\ &= \int \frac{\sec^2(z)}{\sec^2(z)} dz \\ &= \int dz \\ &= z + C\end{aligned}$$

Since $u = \tan(z)$, therefore, $z = \tan^{-1}(u)$. Thus,

$$\begin{aligned}\int \frac{1}{x^2 + 5} dx &= \frac{\sqrt{5}}{5} \cdot \int \frac{1}{u^2 + 1} du \\ &= \frac{\sqrt{5}}{5} \cdot z + C \\ &= \frac{1}{\sqrt{5}} \cdot \tan^{-1}(u) + C\end{aligned}$$

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$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

As a result,

$$\begin{aligned}\int \frac{x^2+x+5}{x^3+5x}dx &= \int \frac{1}{x}dx + \int \frac{1}{x^2+5}dx \\ &= \ln|x| + \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C\end{aligned}$$

Therefore, $\int \frac{x^2+x+5}{x^3+5x}dx = \ln|x| + \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$ ∎

Chap. 7 / Sec. 7.5 / Subsec. 7.5.4 : The denominator $Q(x)$ contains repeated irreducible quadratic factors

Remark 7.5.4. If $Q(x)$ in the rational function $\frac{P(x)}{Q(x)}$ has at least one irreducible quadratic factor, that is, factor of the form,

$$(ax^2+bx+c)^n \qquad \text{where } b^2-4ac<0 \text{ and } n \geq 2$$

them, there exist constants $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ such that the rational function $\frac{P(x)}{Q(x)}$ has a partial fraction decomposition of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(ax^2+bx+c)^n} \\ &= \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}\end{aligned}$$

► **Example 7.5.4.** Evaluate $\int \frac{x^3+x^2+x+1}{x(x^2+1)^2}dx$

Solution. Since x^2+1 cannot be factored any further as $b^2-4ac<0$, therefore we write the integral as,

$$\frac{x^3+x^2+x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Now, we multiply both sides by the denominator $x(x^2+1)^2$ to eliminate the denominators on the right side of the equation. That is,

$$x^3+x^2+x+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

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$$
\begin{aligned}
&= A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x) + Dx^2 + Ex \\
&= (Ax^4 + 2Ax^2 + A) + (Bx^4 + Bx^2 + Cx^3 + Cx) + Dx^2 + Ex \\
&= (Ax^4 + Bx^4) + Cx^3 + (2Ax^2 + Bx^2 + Dx^2) + (Cx + Ex) + A \\
&= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A
\end{aligned}
$$

Now, we compare the coefficients of x^4, x^3, x^2, x, and the constant term both sides of the equation,

$$
\begin{aligned}
A + B &= 0 \\
C &= 1 \\
2A + B + D &= 1 \\
C + E &= 1 \\
A &= 1
\end{aligned}
$$

Solving these equations, we get $A = 1$, $B = -1$, $C = 1$, $D = 0$, and $E = 0$. Thus, we write,

$$
\begin{aligned}
\frac{x^3 + x^2 + x + 1}{x(x^2 + 1)^2} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \\
&= \frac{1}{x} + \frac{-x + 1}{x^2 + 1} + \frac{0 \cdot x + 0}{(x^2 + 1)^2} \\
&= \frac{1}{x} + \frac{1 - x}{x^2 + 1} \\
&= \frac{1}{x} + \frac{1}{x^2 + 1} - \frac{x}{x^2 + 1}
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\int \frac{x^3 + x^2 + x + 1}{x(x^2 + 1)^2} dx &= \int \left(\frac{1}{x} + \frac{1}{x^2 + 1} - \frac{x}{x^2 + 1} \right) dx \\
&= \int \frac{1}{x} dx + \int \frac{1}{x^2 + 1} dx - \int \frac{x}{x^2 + 1} dx
\end{aligned}
$$

Here, $\int \frac{1}{x} dx = \ln|x| + C_1$ and $\int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) + C_2$. However, to integrate $\int \frac{x}{x^2 + 1} dx$, we use the substitution rule and let $u = x^2 + 1$, then $du = 2x dx$ or $\frac{du}{2} = x dx$. Thus, we write,

$$
\int \frac{x}{x^2 + 1} dx = \int \frac{1}{u} \frac{du}{2}
$$

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$$
\begin{aligned}
&= \frac{1}{2} \cdot \int \frac{1}{u} du \\
&= \frac{1}{2} \cdot \ln|u| + C_3 \\
&= \frac{1}{2} \ln\left|x^2+1\right| + C_3
\end{aligned}
$$

Therefore, we write,

$$
\begin{aligned}
\int \frac{x^3+x^2+x+1}{x(x^2+1)^2} dx &= \int \frac{1}{x} dx + \int \frac{1}{x^2+1} dx - \int \frac{x}{x^2+1} dx \\
&= \ln|x| + \tan^{-1}(x) - \frac{1}{2} \ln\left|x^2+1\right| + C
\end{aligned}
$$

Therefore, $\displaystyle\int \frac{x^3+x^2+x+1}{x(x^2+1)^2} dx = \ln|x| - \frac{1}{2} \ln\left|x^2+1\right| + \tan^{-1}(x) + C$ ∎

———————The End of Section 7.5

Partial Fractions

§ 7.6 Improper Integrals

So far, we have discussed the definite integrals where the function being integrated was continuous and the intervals of integration were finite. However, there are situations where the function becomes infinite within the interval of integration or the interval itself extends to infinity. These are called improper integrals.

Chap. 7 / Sec. 7.6 / Subsec. 7.6.1 : Infinite Intervals

Consider the area A under the curve $y = \dfrac{1}{x^2}$ and above the x-axis, over the interval $[1, \infty)$, as shown below.

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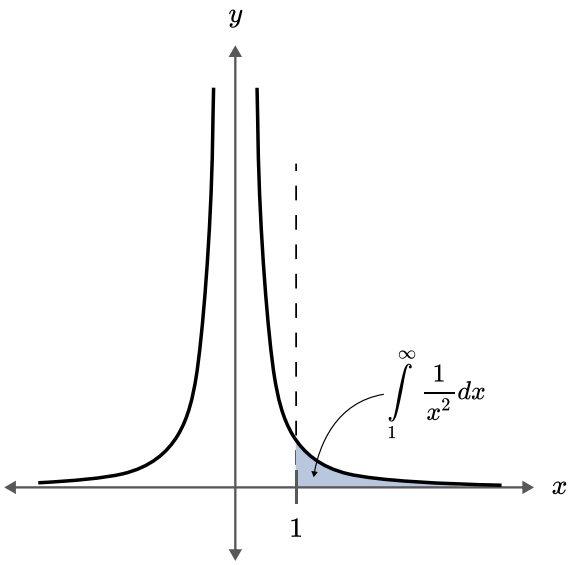

Figure 7.4

Since the interval of integration is infinite, therefore it may be intuitive to say that the area A would also be infinite. However, to our surprise, this is not the case. This can be tested. For example, if we let $x = b$ be a line such that $b > 1$, then the area between $x = 1$ and $x = b$ yields,

$$\begin{aligned} A &= \int_1^b \frac{1}{x^2}dx \\ &= \int_1^b x^{-2}dx \\ &= \frac{x^{-2+1}}{-2+1}\Big]_1^b \\ &= -\frac{1}{x}\Big]_1^b \\ \therefore A &= 1 - \frac{1}{b} \end{aligned}$$

Now, notice that the area A over the finite interval $[1, b]$ will always be less than 1 for all $b > 1$ because of the term $\frac{1}{b}$. If we let $b \to \infty$ to find the area A over the interval $[1, \infty)$, then

$$\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1 - 0 = 1$$

That is, the area $A \to 1$ as $b \to \infty$. Thus, the area A under the curve $y = \frac{1}{x^2}$, above the x-axis and right to the vertical line $x = 1$ is 1, and we write,

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$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$$
$$= 1$$

Based on that observation, we now define the following.

Definition 7.6.1. (Infinite Intervals)

1. If $f(x)$ is continuous over $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$
given that the limit exists.

2. If $f(x)$ is continuous over $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$
given that the limit exists.

3. If $f(x)$ is continuous over $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$
given that the both limit exists.

In each case, if the limit exists, then the improper integral is called *convergent*, if not, it is called *divergent*.

► **Example 7.6.1.** Is the integral $\int_1^\infty \frac{1}{x}dx$ convergent or divergent?

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Solution. From the definition above, we write,

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}dx\\ &= \lim_{b\to\infty}\left(\ln(x)\Big]_1^b\right)\\ &= \lim_{b\to\infty}\left(\ln(b)-\ln(1)\right)\\ &= \lim_{b\to\infty}\ln(b)\\ &= \infty\end{aligned}$$

Since the limit does not yield a finite value, therefore the limit does not exist. As a result, the integral $\int_1^\infty \frac{1}{x}dx$ is divergent. ∎

► **Example 7.6.2.** Evaluate $\int_0^\infty \frac{1}{1+x^2}dx$

Solution. According to the definition of improper integrals, we write,

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx\\ &= \lim_{b\to\infty}\tan^{-1}(x)\Big]_0^b\\ &= \lim_{b\to\infty}\left(\tan^{-1}(b)-\tan^{-1}(0)\right)\\ &= \lim_{b\to\infty}\tan^{-1}(b)\\ &= \frac{\pi}{2}\end{aligned}$$

Therefore, $\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$ ∎

Chap. 7 / Sec. 7.6 / Subsec. 7.6.2 : Discontinuous Integrands

In the previous subsection, we observed that the area of the region extended infinitely horizontally because the interval of integration was infinite. In this

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subsection, we will examine how the area of the region extends infinitely in the vertical direction. To illustrate, consider the area A under the curve $y = \dfrac{1}{\sqrt{x}}$ and above the x-axis, from $x = 0$ to $x = 1$, as shown below.

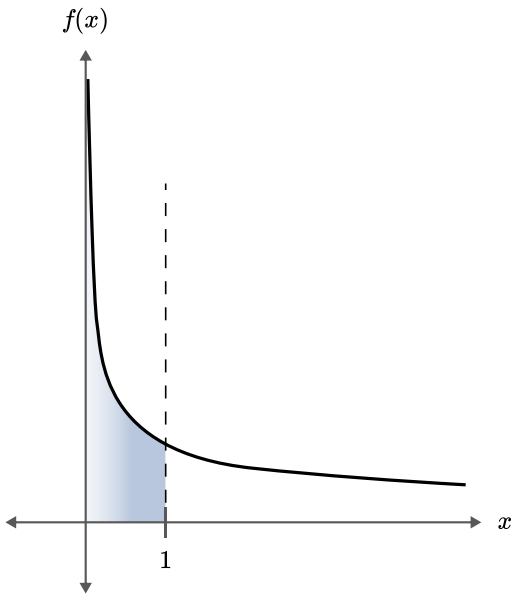

Figure 7.5

Notice that $y = \dfrac{1}{\sqrt{x}}$ increases without a bound as $x \to 0^+$, and is not defined at the exact point $x = 0$. However, we can still determine the area $A = \displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$, if we first determine the integral $\displaystyle\int_a^1 \frac{1}{\sqrt{x}}dx$ and take the limit as $a \to 0^+$. That is,

$$\begin{aligned}\int_a^1 \frac{1}{\sqrt{x}}dx &= 2\sqrt{x}\Big]_a^1 \\ &= 2\sqrt{1} - 2\sqrt{a} \\ &= 2 - 2\sqrt{a} \\ &= 2\left(1 - \sqrt{a}\right)\end{aligned}$$

Now, take the limit of $2\left(1 - \sqrt{a}\right)$ as $a \to 0^+$.

$$\begin{aligned}\lim_{a\to 0^+} 2\left(1 - \sqrt{a}\right) &= 2\left(1 - 0\right) \\ &= 2\end{aligned}$$

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Observe how the area of the bounded region is finite, even though the length of the boundary curve is infinite!

Based on the observation above, we define the following.

Definition 7.6.2. (Discontinuous Integrands)

1. If $f(x)$ is continuous over $(a, b]$, where $\lim_{x\to a^+} f(x) = \pm\infty$. Then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$
given that the limit exists.

2. If $f(x)$ is continuous over $[a, b)$, where $\lim_{x\to b^-} f(x) = \pm\infty$. Then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$
given that the limit exists.

3. If $f(x)$ is continuous over $[a, b]$, except at a point c such that $a < c < b$. Then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
given that improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist.

In every case, if the limits exist, the improper integral is called *convergent*, if not, it is called *divergent*.

► **Example 7.6.3.** Evaluate $\int_0^1 \ln(x)dx$

Solution. Since $\ln(x)$ has a vertical asymptote at the point $x = 0$, therefore we

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write the following,

$$\int_0^1 \ln(x)dx = \lim_{a\to 0^+} \int_a^1 \ln(x)dx$$

Since we know that,

$$\int \ln(x)dx = x\ln(x) - x + C$$

Therefore, we write the following,

$$\begin{aligned}
\lim_{a\to 0^+} \int_a^1 \ln(x)dx &= \lim_{a\to 0^+} \Big[x\ln(x) - x\Big]_a^1 \\
&= \lim_{a\to 0^+} ((1\cdot\ln(1) - 1) - (a\ln(a) - a)) \\
&= \lim_{a\to 0^+} (\ln(1) - 1 - a\ln(a) + a) \\
&= \lim_{a\to 0^+} (a - a\ln(a) - 1) \\
&= \lim_{a\to 0^+} a - \lim_{a\to 0^+} a\ln(a) - \lim_{a\to 0^+} 1 \\
&= 0 - \lim_{a\to 0^+} a\ln(a) - 1
\end{aligned}$$

Now, to find the limit $\lim\limits_{a\to 0^+} a\ln(a)$, we apply the L'Hôpital's Rule as follows,

$$\begin{aligned}
\lim_{a\to 0^+} a\ln(a) &= \lim_{a\to 0^+} \frac{\ln(a)}{\frac{1}{a}} \\
&= \lim_{a\to 0^+} \frac{\frac{d}{da}\ln(a)}{\frac{d}{da}\frac{1}{a}} \\
&= \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\
&= \lim_{a\to 0^+} (-a) \\
&= 0
\end{aligned}$$

As a result, we get,

$$\int_0^1 \ln(x)dx = \lim_{a\to 0^+} \int_a^1 \ln(x)dx$$

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$$
\begin{aligned}
&= 0 - \lim_{a \to 0^+} a \ln(a) - 1 \\
&= 0 - 0 - 1 \\
&= -1
\end{aligned}
$$

Therefore, $\int_0^1 \ln(x)dx = -1$ ∎

———————The End of Section 7.6
Improper Integrals

§ 7.7 Table of Integrals

Differentiation has some straightforward rules for finding the derivative of a complicated function. Integration, however, lacks such straightforwardness. As a result, tables of known integrals often prove helpful to aid in finding antiderivatives.

In integration, finding the antiderivative does not follow a set of simple, universal rules. Integrating complex functions directly is often challenging and less intuitive. For this reason, mathematicians have compiled tables of integrals. These tables list the integrals of many common functions. By referencing these tables, we can quickly find the antiderivative of a function that matches one of the known forms, without having to derive each one from scratch.

Author's Interruption. While most textbook authors place the table of integrals in the appendices to serve as a reference and avoid distracting the reader, I have chosen to include them in the main text. This is due to the reason I felt that the integrals listed in the table are often overlooked by readers. Although memorizing every single integral listed in the table is not necessary at a beginner to intermediate level, I believe that a thoughtful review of them can be valuable for those interested in advanced calculus or mathematical research. For beginners, the elementary integrals should be of top priority.

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Chap. 7 / Sec. 7.7 / Subsec. 7.7.1 : Elementary Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$
2. $\int dx = x + c$
3. $\int \frac{dx}{x} = \int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$
4. $\int e^{ax} dx = \frac{e^{ax}}{a} + C \qquad a \neq 0$
5. $\int b^{ax} dx = \frac{b^{ax}}{a \ln(b)} + C \qquad b > 0, \quad b \neq 1, \quad a \neq 0$
6. $\int \ln(x)\, dx = x \ln(x) - x + C$
7. $\int \log_b(x)\, dx = \frac{1}{\ln(b)}\Big(x \ln(x) - x\Big) + C \qquad b > 0, \quad b \neq 1$
8. $\int \sin(x)\, dx = -\cos(x) + C$
9. $\int \cos(x)\, dx = \sin(x) + C$
10. $\int \tan(x)\, dx = -\ln\Big|\cos(x)\Big| + C$
11. $\int \cot(x)\, dx = \ln\Big|\sin(x)\Big| + C$
12. $\int \sec(x)\, dx = \ln\Big|\tan(x) + \sec(x)\Big| + C$
13. $\int \csc(x)\, dx = -\ln\Big|\cot(x) + \csc(x)\Big| + C$
14. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad a > 0, \quad |x| < 0$
15. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad a > 0$
16. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \qquad a > 0, \quad |x| > 0$

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Chap. 7 / Sec. 7.7 / Subsec. 7.7.2 : Trigonometric Integrals

1. $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C \qquad a \neq 0$
2. $\int \sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) + C$
3. $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C \qquad a \neq 0$
4. $\int \cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax) + C$
5. $\int \tan(ax)\,dx = -\frac{1}{a}\ln\left|\cos(ax)\right| + C \qquad a \neq 0$
6. $\int \tan^2(ax)dx = \frac{1}{a}\tan(ax) - x + C$
7. $\int \cot(ax)\,dx = \frac{1}{a}\ln\left|\sin(ax)\right| + C \qquad a \neq 0$
8. $\int \cot^2(ax)dx = -\frac{1}{a}\cot(ax) - x + C$
9. $\int \sec(ax)\,dx = \frac{1}{a}\ln\left|\tan(ax) + \sec(ax)\right| + C \qquad a \neq 0$
10. $\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C$
11. $\int \csc(ax)\,dx = -\frac{1}{a}\ln\left|\cot(ax) + \csc(ax)\right| + C \qquad a \neq 0$
12. $\int \csc^2(ax)dx = -\frac{1}{a}\cot(ax) + C$
13. $\int x\sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$
14. $\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$
15. $\int x^2\sin^2(ax)dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + C$
16. $\int x^2\cos^2(ax)dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin(2ax) + \frac{x}{4a^2}\cos(2ax) + C$

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Chap. 7 / Sec. 7.7 / Subsec. 7.7.3 : Inverse Trigonometric Integrals

1. $\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$
2. $\int \sin^{-1}(ax)dx = x\sin^{-1}(ax) + \dfrac{\sqrt{1-a^2x^2}}{a} + C$
3. $\int \cos^{-1}(x)\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C$
4. $\int \cos^{-1}(ax)dx = x\cos^{-1}(ax) - \dfrac{\sqrt{1-a^2x^2}}{a} + C$
5. $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \dfrac{\ln\left(1+x^2\right)}{2} + C$
6. $\int \tan^{-1}(ax)\,dx = x\tan^{-1}(ax) - \dfrac{\ln\left(1+a^2x^2\right)}{2a} + C$
7. $\int \cot^{-1}(x)\,dx = x\cot^{-1}(x) + \dfrac{\ln\left(1+x^2\right)}{2} + C$
8. $\int \cot^{-1}(ax)\,dx = x\cot^{-1}(ax) + \dfrac{\ln\left(1+a^2x^2\right)}{2a} + C$
9. $\int \sec^{-1}(x)\,dx = x\sec^{-1}(x) - \ln\left(x+\sqrt{x^2-1}\right) + C$
10. $\int \sec^{-1}(ax)\,dx = x\sec^{-1}(ax) - \dfrac{1}{a}\ln\left(ax+\sqrt{a^2x^2-1}\right) + C$
11. $\int \csc^{-1}(x)\,dx = x\csc^{-1}(x) + \ln\left(x+\sqrt{x^2-1}\right) + C$
12. $\int \csc^{-1}(ax)\,dx = x\csc^{-1}(ax) + \dfrac{1}{a}\ln\left(ax+\sqrt{a^2x^2-1}\right) + C$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.4 : Reduction Formula for Trigonometric Functions

1. $\int \sin^n(x)\,dx = -\dfrac{\sin^{n-1}(x)\cos(x)}{n} + \dfrac{n-1}{n}\int \sin^{n-2}(x)\,dx$
2. $\int \cos^n(x)\,dx = \dfrac{\cos^{n-1}(x)\sin(x)}{n} + \dfrac{n-1}{n}\int \cos^{n-2}(x)\,dx$

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3. $\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx, \quad n \neq 1$

4. $\int \cot^n(x)\,dx = -\frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x)\,dx, \quad n \neq 1$

5. $\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} - \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx, \quad n \neq 1$

6. $\int \csc^n(x)\,dx = \frac{\csc^{n-2}(x)\cot(x)}{n-1} - \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx, \quad n \neq 1$

7. $\int \sin^m(x)\cos^n(x)\,dx = \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)\,dx,$
$m \neq -n$

8. $\int x^n \sin(ax)\,dx = -\frac{x^n\cos(ax)}{a} + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx, \quad a \neq 0$

9. $\int x^n \cos(ax)\,dx = \frac{x^n\sin(ax)}{a} - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx, \quad a \neq 0$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.5 : Integrals of the Form $a^2 - x^2, a > 0$

1. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

2. $\int x^2\sqrt{a^2-x^2}dx = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) + C$

3. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

4. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

5. $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

6. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{1}{x}\sqrt{a^2-x^2} - \sin^{-1}\left(\frac{x}{a}\right) + C$

7. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

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Chap. 7 / Sec. 7.7 / Subsec. 7.7.6 : Integrals of the Form $x^2 - a^2, a > 0$

1. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + C$
2. $\int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} \left(2x^2 - a^2\right) \sqrt{x^2 - a^2} - \frac{a^4}{8} \ln \left| x + \sqrt{x^2 - a^2} \right| + C$
3. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$
4. $\int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$
5. $\int \frac{x^2}{\sqrt{x^2 - a^2}} dx = \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + \frac{x}{2}\sqrt{x^2 - a^2} + C$
6. $\int \frac{\sqrt{x^2 - a^2}}{x^2} dx = \ln \left| x + \sqrt{x^2 - a^2} \right| - \frac{\sqrt{x^2 - a^2}}{x} + C$
7. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$
8. $\int \frac{dx}{x\left(x^2 - a^2\right)} = \frac{1}{2a^2} \ln \left| \frac{x^2 - a^2}{x^2} \right| + C$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.7 : Integrals of the Form $a^2 + x^2, a > 0$

1. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
2. $\int x^2 \sqrt{a^2 + x^2} dx = \frac{x}{8} \left(a^2 + 2x^2\right) \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
3. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
4. $\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a} \ln \left(a - \sqrt{a^2 + x^2} \right) + C$
5. $\int \frac{dx}{x^2 \sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C$
6. $\int \frac{x^2}{\sqrt{a^2 + x^2}} dx = -\frac{a^2}{2} \ln \left(x + \sqrt{a^2 + x^2} \right) + \frac{x}{2}\sqrt{a^2 + x^2} + C$

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7. $\int \frac{\sqrt{a^2+x^2}}{x}dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

8. $\int \frac{\sqrt{a^2+x^2}}{x^2}dx = \ln\left|x+\sqrt{a^2+x^2}\right| - \frac{\sqrt{a^2+x^2}}{x} + C$

9. $\int \frac{dx}{\sqrt{(a^2+x^2)^3}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$

10. $\int \frac{dx}{x\left(a^2+x^2\right)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right) + C$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.8 : Integrals of the Form $ax \pm b, a \neq 0,\ b > 0$

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C, \qquad n \neq -1$

2. $\int \left(\sqrt{ax+b}\right)^n dx = \frac{2\left(\sqrt{ax+b}\right)^{n+2}}{a\,(n+2)} + C, \qquad n \neq -2$

3. $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

4. $\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\left(\sqrt{\frac{ax-b}{b}}\right) + C$

5. $\int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b\ln|ax+b|}{a^2} + C$

6. $\int \frac{x^2}{ax+b}dx = \frac{(ax+b)^2 - 4b\,(ax+b) + 2b^2\ln|ax+b|}{2a^3} + C$

7. $\int \frac{dx}{x\,(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

8. $\int \frac{dx}{x^2\,(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$

9. $\int \frac{x}{\sqrt{ax+b}}dx = \frac{2\,(ax-2b)\sqrt{ax+b}}{3a^2} + C$

10. $\int x\sqrt{ax+b}dx = \frac{2}{15a^2}\,(3ax-2b)\sqrt{(ax+b)^3} + C$

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11. $\int x(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a^2}\left(\frac{ax+b}{n+2} - \frac{b}{n+1}\right) + C, \qquad n \neq -1, -2$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.9 : Integrals with Exponential and Trigonometric Functions

1. $\int e^{ax} \sin(bx)\, dx = \frac{e^{ax}(a\sin(bx) - b\cos(bx))}{a^2+b^2} + C$
2. $\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}(a\cos(bx) - b\sin(bx))}{a^2+b^2} + C$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.10 : Integrals with Exponential and Logarithmic Functions

1. $\int \frac{dx}{x\ln(x)} = \ln|\ln(x)| + C$
2. $\int x^n \ln(x)\, dx = \frac{x^{n+1}\left(\ln(x) - \frac{1}{n+1}\right)}{n+1} + C, n \neq -1$
3. $\int xe^x dx = xe^x - e^x + C$
4. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx, a \neq 0$
5. $\int \ln^n(x)\, dx = x\ln^n(x) - n\int \ln^{n-1}(x)\, dx$

Chap. 7 / Sec. 7.7 / Subsec. 7.7.11 : Miscellaneous Integral Forms

1. $\int x^n \sin^{-1}(x)\, dx = \frac{x^{n+1}\sin^{-1}(x) - \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx}{n+1}, n \neq -1$
2. $\int x^n \cos^{-1}(x)\, dx = \frac{x^{n+1}\cos^{-1}(x) + \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx}{n+1}, n \neq -1$

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3. $\int x^n \tan^{-1}(x)\,dx = \dfrac{x^{n+1}\tan^{-1}(x) - \displaystyle\int \frac{x^{n+1}}{x^2+1}dx}{n+1}, n \neq -1$

4. $\int \sqrt{2ax - x^2}dx = \dfrac{(x-a)\sqrt{2ax-x^2}}{2} + \dfrac{a^2 \sin^{-1}\left(\dfrac{x-a}{a}\right)}{2} + C, a > 0$

5. $\int \dfrac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\dfrac{x-a}{a}\right) + C, a > 0$

———————The End of Section 7.7

Table of Integrals

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Chapter 8

Applications of Integration

In the previous chapters, definite integrals were discussed in terms of the Riemann Sum, and it was shown how they can be evaluated using antiderivatives through the Fundamental Theorem of Calculus. This chapter will now focus on the definite integral as a computational tool and demonstrate its importance in solving and understanding a wide range of mathematical problems.

§ 8.1 Average Value of a Function

In mathematics, a fundamental challenge when dealing with a continuous function defined over an interval is determining the average of its values throughout that interval. Unlike discrete data, where we can calculate the average by summing a finite number of values and dividing it by the count, continuous functions pose a different challenge. A function on a closed interval, say $[a, b]$, has infinitely many values between a and b, so the traditional notion of averaging cannot directly apply.

To handle this, we define the average value of a function in terms of an integral. For this, we let $f(x)$ be a continuous function defined over an interval $[a, b]$. We divide the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = \dfrac{b-a}{n}$. Let x_i^* be a sample point in each sub-interval $[x_{i-1}, x_i]$ and calculate $f(x_i^*)$ for $i = 1, 2, 3, \cdots, n$. Then,

$$f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)$$

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is the total value of the function. Now, to take its average, we divide it by the number of terms, that is, divide it by n,

$$\frac{f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)}{n}$$

Now, since $\Delta x = \dfrac{b-a}{n}$, therefore $n = \dfrac{b-a}{\Delta x}$, and thus,

$$\frac{f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}}$$

Since, $f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)$ can be concisely written as $\displaystyle\sum_{i=1}^{n} f(x_i^*)$, therefore,

$$\begin{aligned}
\frac{f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} &= \frac{\displaystyle\sum_{i=1}^{n} f(x_i^*)}{\dfrac{b-a}{\Delta x}} \\
&= \frac{1}{\dfrac{b-a}{\Delta x}} \cdot \sum_{i=1}^{n} f(x_i^*) \\
&= \frac{\Delta x}{b-a} \cdot \sum_{i=1}^{n} f(x_i^*) \\
&= \frac{1}{b-a} \left(\Delta x \cdot \sum_{i=1}^{n} f(x_i^*) \right) \\
&= \frac{1}{b-a} \cdot \underbrace{\sum_{i=1}^{n} f(x_i^*) \Delta x}_{\text{Riemann sum}}
\end{aligned}$$

Now, taking $n \to \infty$ to get the exact value, we get,

$$\begin{aligned}
&= \frac{1}{b-a} \cdot \left(\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \right) \\
&= \frac{1}{b-a} \cdot \left(\int_a^b f(x)\, dx \right)
\end{aligned}$$

This is the average value of a function. We denote this as f_{ave}

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Definition 8.1.1. (The Average Value of a Function)
Let $f(x)$ be a continuous function over an interval $[a, b]$, then the average value of $f(x)$, denoted as f_{ave}, is defined as,

$$f_{\text{ave}} = \frac{1}{b-a} \cdot \int_a^b f(x)\, dx$$

A few examples are shown below to demonstrate the definition above.

► **Example 8.1.1.** Find the average value of the function $f(x) = x^2$ over the interval $[0, 2]$.

Solution. Since $f(x) = x^2$, therefore $\int x^2 dx = \frac{x^3}{3} + C$. Thus, we write,

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-0} \cdot \int_0^2 x^2 dx \\ &= \frac{1}{2} \cdot \frac{x^3}{3}\Big]_0^2 \\ &= \frac{1}{2} \cdot \frac{2^3}{3} \\ &= \frac{4}{3} \end{aligned}$$

Therefore, the average value of the function $f(x) = x^2$ over the interval $[0, 2]$ is $\frac{4}{3}$. ∎

► **Example 8.1.2.** Find the average value of the function $f(x) = e^x$ over the interval $[0, e]$.

Solution. Since $f(x) = e^x$, therefore $\int e^x dx = e^x + C$. Thus, we write,

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{e-0} \cdot \int_0^e e^x dx \\ &= \frac{1}{e} \cdot e^x\Big]_0^e \\ &= \frac{1}{e} \cdot \left(e^e - e^0\right) \end{aligned}$$

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$$= \frac{e^e - 1}{e}$$
$$\approx 5.207$$

Therefore, the average value of the function $f(x) = e^x$ over the interval $[0, e]$ is approximately 5.207. ∎

Chap. 8 / Sec. 8.1 / Subsec. 8.1.1 : The Mean Value Theorem for Integrals

When we were discussing the average value of a function over a specific interval, we might have wondered if there was a particular point within that interval where the function's value is exactly equal to its average value over the entire interval, as the function can take on infinitely many values. That is, if there exists a specific point, say c, within the interval such that $f(c)$ is the same as the average values of the function f_{ave} within that range. This observation brings us to the mean value theorem for integrals.

Theorem 8.1.1. (The Mean Value Theorem For Integrals) If $f(x)$ is a continuous function over an interval $[a, b]$, then there exists a point $c \in [a, b]$, such that

$$\begin{aligned} f(c) &= f_{\text{ave}} \\ &= \frac{1}{b-a} \cdot \int_a^b f(x)\, dx \end{aligned}$$

Equivalently,

$$\int_a^b f(x)\, dx = f(c) \cdot (b - a)$$

A few examples to demonstrate the theorem above are provided below.

► **Example 8.1.3.** Find a number c in the interval $[0, 2]$ such that the average value of $f(x) = x + 2$ equals $f(c)$.

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Solution. First, we will find the average value of $f(x) = x + 2$ on $[0, 2]$ as follows,

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-0} \cdot \int_0^2 (x+2)dx \\ &= \frac{1}{2} \cdot \left[\frac{x^2}{2} + 2x\right]_0^2 \\ &= \frac{1}{2} \cdot \left(\frac{2^2}{2} + 2 \cdot 2\right) \\ &= 3 \end{aligned}$$

Now, according to the mean value theorem for integrals, there must be a point $c \in [0, 2]$ such that $f(c) = 3$. That is,

$$\begin{aligned} f(c) = c + 2 &= 3 \\ c &= 3 - 2 \\ &= 1 \end{aligned}$$

Therefore, at $c = 1$, the function reaches its average value of 3. ∎

► **Example 8.1.4.** Find a number c in the interval $[0, 1]$ such that the average value of $f(x) = e^x$ equals $f(c)$.

Solution. First, we will find the average value of $f(x) = e^x$ on $[0, 1]$ as follows,

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{1-0} \cdot \int_0^1 e^x dx \\ &= e^x \Big]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \end{aligned}$$

Now, according to the mean value theorem for integrals, there must be a point $c \in [0, 1]$ such that $f(c) = e - 1$. That is,

$$\begin{aligned} f(c) = e^c &= e - 1 \\ \ln(e^c) &= \ln(e-1) \\ c \cdot \underbrace{\ln(e)}_{1} &= \ln(e-1) \end{aligned}$$

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$$c = \ln(e-1)$$

Therefore, at $c = \ln(e-1)$, the function reaches its average value of $e-1$. ∎

THE END OF SECTION 8.1
Average Value of a Function

§ 8.2 Area of a Region Between Two Curves

The idea of finding the area of a region enclosed between two curves is essentially an extension of the basic idea of calculating the area under a curve. For example, consider two functions $f(x)$ and $g(x)$ continuous over an interval $[a,b]$, where $f(x) \geq g(x)$.

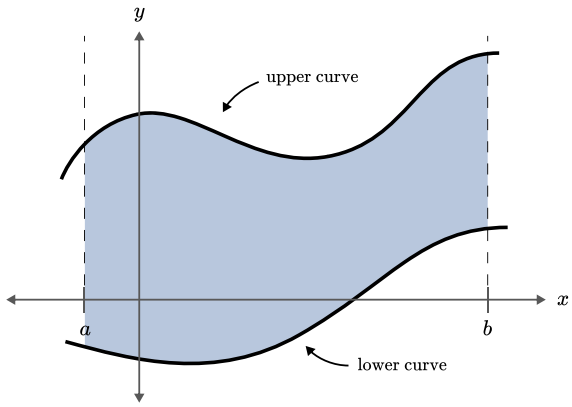

Figure 8.1

Now, in order to find the area of the region that is enclosed by the curves over the interval $[a,b]$, we partition the interval $[a,b]$ into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$. On each sub-interval, we construct a rectangle spanning from the lower curve to the upper curve. Thus, the height of the ith rectangle yields $f(x_i^*) - g(x_i^*)$, where, x_i^* is a sample point chosen from the ith sub-interval. Therefore, the area of the ith rectangle yields $\underbrace{(f(x_i^*) - g(x_i^*))}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$. Summing the areas of the n rectangles provides an estimate for the exact area enclosed by the curves, that is,

$$A \approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \cdot \Delta x$$

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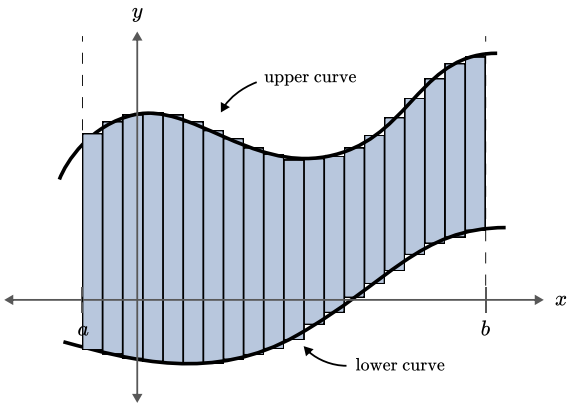

Figure 8.2

This forms the Riemann sum. Now, letting $n \to \infty$, we get the exact area A bounded by the curves and the vertical lines $x = a$ and $x = b$. That is,

$$A = \lim_{x \to \infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \cdot \Delta x$$

As a result, we have the following definition of the area of a region between two curves.

Definition 8.2.1. (Area of a Region Between Two Curves) Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over the interval $[a, b]$. The area of the region bounded by the graphs of $f(x)$ and $g(x)$ over $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))dx$$

A few examples to demonstrate the definition above are shown below.

► **Example 8.2.1.** Find the area of the region bounded by $f(x) = \sin(x)$ and $g(x) = \cos(x)$, over the interval $[1, 3]$.

Solution. The graph of $f(x) = \sin(x)$ is the upper curve and the graph of $g(x) = \cos(x)$ is the lower curve on the interval $[1, 3]$. Therefore, the area of the region between these curves is,

$$\int_1^3 (\sin(x) - \cos(x))\, dx = \int_1^3 \sin(x)dx - \int_1^3 \cos(x)dx$$

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$$
\begin{aligned}
&= \int_1^3 \sin(x)dx - \int_1^3 \cos(x)dx \\
&= -\cos(x)\Big]_1^3 - \sin(x)\Big]_1^3 \\
&= -\cos(3) + \cos(1) - \sin(3) + \sin(1) \\
&\approx 2.23
\end{aligned}
$$

Therefore, the area of the region bounded by $f(x) = \sin(x)$ and $g(x) = \cos(x)$, over the interval $[1, 3]$ is approximately 2.23. ∎

► **Example 8.2.2.** Find the area of the region bounded by $f(x) = x$ and $g(x) = x^2$, over the interval $[0, 1]$.

Solution. The graph of $f(x) = x$ is the upper curve and the graph of $g(x) = x^2$ is the lower curve on the interval $[0, 1]$. Therefore, the area of the region between these curves is,

$$
\begin{aligned}
\int_0^1 (x - x^2)dx &= \int_0^1 xdx - \int_0^1 x^2dx \\
&= \frac{x^2}{2}\Big]_0^1 - \frac{x^3}{3}\Big]_0^1 \\
&= \frac{1}{2} - \frac{1}{3} \\
&\approx 0.167
\end{aligned}
$$

Therefore, the area of the region bounded by $f(x) = x$ and $g(x) = x^2$, over the interval $[0, 1]$ is approximately 0.167. ∎

Chap. 8 / Sec. 8.2 / Subsec. 8.2.1 : Area of a Region between Two Curves with respect to y

There are times when it is easier to integrate the area with respect to y instead of x. For example, consider integrating the area of a region enclosed by the graph of $x = f(y)$ and $x = g(y)$ over the interval $[c, d]$ lying on the y-axis, where $f(y) \geq g(y)$, implying that the graph of $f(y)$ lies to the right of the graph of $g(y)$

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and the lower and upper bound of the region are $y = c$ and $y = d$, respectively, as shown below.

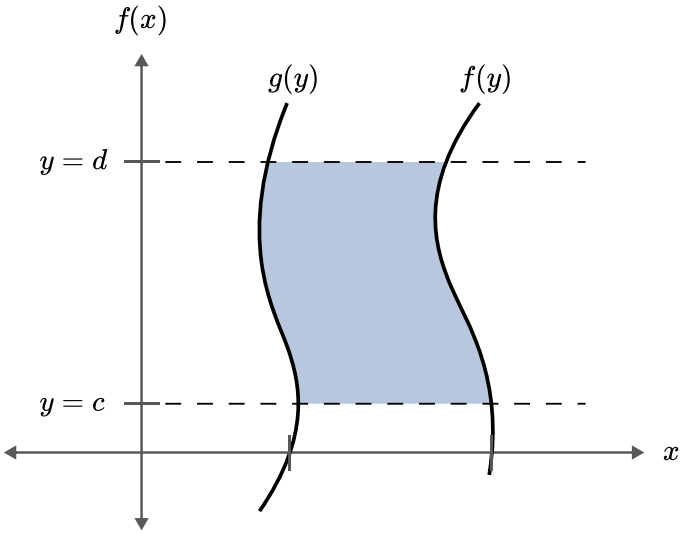

Figure 8.3

Here, in order to integrate, we treat the dependent variable y as the independent variable and divide the interval $[c, d]$ into n sub-interval of equal height $\Delta y = \dfrac{d-c}{n}$. On each sub-interval, we construct a rectangle spanning from the left curve to the right curve. Thus, the width of the ith rectangle yields $f(y_i^*) - g(y_i^*)$, where y_i^* is a sample point chosen from the ith sub-interval. Therefore, the area of the ith rectangle yields $\underbrace{f(y_i^*) - g(y_i^*)}_{\text{width}} \cdot \underbrace{\Delta y}_{\text{height}}$. Summing the areas of the n rectangles and taking the limit as $n \to \infty$ provides the exact area of a region enclosed between two curves with respect to y, that is,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f\left(y_i^*\right) - g\left(y_i^*\right)\right) \Delta y$$

As a result, we have the following definition of the area of a region enclosed between two curves with respect to the y-axis.

Definition 8.2.2 (Area of a Region Between Two Curves with respect to y). Let $f(y)$ and $g(y)$ be continuous functions such that $f(y) \geq g(y)$ over the interval $[c, d]$. The area of the region bounded by the graphs of $f(y)$ and $g(y)$ over

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$[c, d]$ is

$$A = \int_c^d (f(y) - g(y))dy$$

A few examples to demonstrate the definition above are provided below.

► **Example 8.2.3.** Find the area of the region bounded by $x = y^2$ and $x = 4y$.

Solution. In order to integrate, we must first identify the integration intervals by finding the points of intersection of the two functions, which is done by equating the functions as shown below.

$$\begin{aligned} y^2 &= 4y \\ y^2 - 4y &= 0 \\ y(y-4) &= 0 \end{aligned}$$

Therefore, the points of intersection are $y = 0$ and $y = 4$.

Since $x = 4y$ is to the right of $x = y^2$ in that region, therefore, we have the following,

$$\begin{aligned} \int_0^4 \left(4y - y^2\right) dy &= \int_0^4 4ydy - \int_0^4 y^2 dy \\ &= 4\int_0^4 ydy - \int_0^4 y^2 dy \\ &= 4\left(\frac{y^2}{2}\Big]_0^4\right) - \frac{y^3}{3}\Big]_0^4 \\ &= 4\left(\frac{4^2}{2}\right) - \frac{4^3}{3} \\ &= \frac{32}{3} \end{aligned}$$

Therefore, the area of the region bounded by $x = y^2$ and $x = 4y$ is $\frac{32}{3}$. ∎

► **Example 8.2.4.** Find the area of the region bounded by the graph of $y = x^2$ and $y = x + 2$.

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Solution. We will integrate the area of the bounded region with respect to y. Therefore, we first need to solve for x in terms of y, as shown below.

$$\begin{aligned} y = x^2 \qquad &\Longrightarrow \qquad x = \sqrt{y} \\ y = x + 2 \qquad &\Longrightarrow \qquad x = y - 2 \end{aligned}$$

Now, to find the limits of integration, we set $x^2 = x + 2$ as follows,

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \end{aligned}$$

Therefore, the points of intersection are $x = 2$ and $x = -1$.
Now, we substitute these x-values into the original functions to find the corresponding y-values, as shown below.

$$\begin{aligned} x = 2 \qquad &\Longrightarrow \qquad y = 2^2 = 4 \\ x = -1 \qquad &\Longrightarrow \qquad y = (-1) + 2 = 1 \end{aligned}$$

Now, notice that since $x = y - 2$ is to the right of $x = \sqrt{y}$, therefore, we have the following,

$$\begin{aligned} \int_1^4 \Big(\sqrt{y} - (y-2)\Big)dy &= \int_1^4 \sqrt{y}dy - \int_1^4 (y-2)dy \\ &= \int_1^4 \sqrt{y}dy - \left(\int_1^4 ydy - \int_1^4 2dy\right) \\ &= \int_1^4 y^{\frac{1}{2}}dy - \int_1^4 ydy + 2\int_1^4 dy \\ &= \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_1^4 - \frac{y^2}{2}\Bigg]_1^4 + 2\left(y\right]_1^4\Big) \\ &= \left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}}\right) - \left(\frac{4^2}{2} - \frac{1^2}{2}\right) + 2\,(4-1) \\ &= \left(\frac{8}{\frac{3}{2}} - \frac{1}{\frac{3}{2}}\right) - \left(8 - \frac{1}{2}\right) + 6 \end{aligned}$$

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$$= \frac{19}{6}$$

Therefore, the area of the region bounded by the graph of $y = x^2$ and $y = x + 2$ is $\frac{19}{6}$. ∎

———————The End of Section 8.2

Area of a Region Between Two Curves

§ 8.3 Volume: The Slicing Method

Area, as we intuitively understand it, is the amount of space that a two-dimensional shape occupies. For example, the area of a rectangle is calculated as the product of its length and width, quantifying the amount of space it covers. Extending this concept to the three dimensions introduces the idea of "volume." That is, volume is the amount of space that a three-dimensional object occupies. For example, the volume of a rectangular box is found by multiplying its length, width, and height, or equivalently, multiplying the area of its base by its height.

Although intuitive, these ideas are too vague to serve as a formal definition, as they lack the necessary level of mathematical rigor. Nevertheless, having previously defined the concept of area rigorously in terms of the definite integral, we can now use that definition to define the concept of volume and discuss methods for calculating a specific type of volume: the volume of a solid of revolution. In order to do that, three primary techniques will be discussed in this chapter—slicing, disks, and washers—each appropriate for solids with particular characteristics.

Having this on mind, we now begin our discussion of defining the volume of a solid.

Chap. 8 / Sec. 8.3 / Subsec. 8.3.1 : The Definition of the Volume of a Solid

To precisely define the volume of a solid, we make use of the slicing method. This is a technique that determines the volume by integrating the cross-sectional areas of the solid. Put simply: this method works by "slicing" the solid into thin cross-sectional pieces, calculating the *area* of each slice, and then summing up

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these areas across the entire volume.

A natural question is now to follow—what is a cross-section? Informally speaking, a cross-section is essentially a slice of the solid, which has a certain shape depending on the solid itself. For example, consider the following solid S of a right circular cylinder.

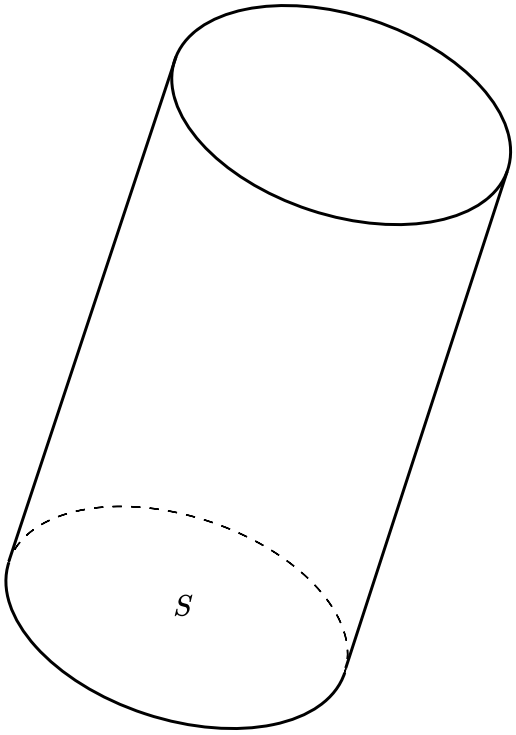

Figure 8.4

When this solid S is intersected with a two-dimensional plane P, the resulting region (colored in blue) is called the cross-section of the solid S, as shown below.

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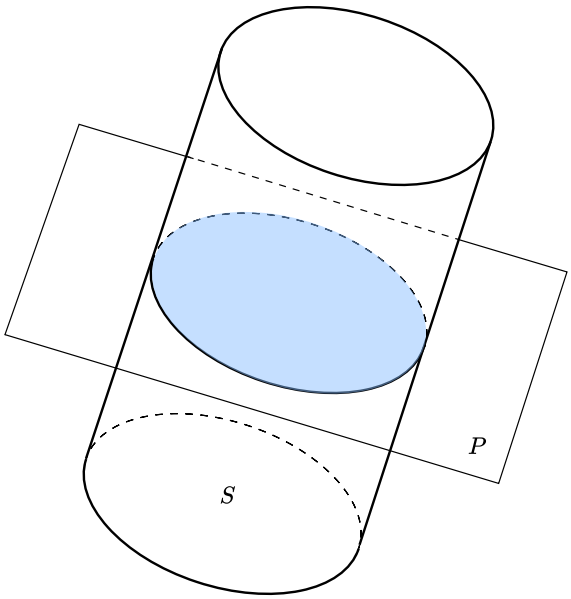

Figure 8.5

Here, the cross-section turns out to be a circle, because the solid S is a right circular cylinder. The area of this cross-section is what we refer to as the "cross-sectional area" (the region highlighted in blue, in this case).

We can use this cross-sectional area to define the volume of the solid S. The idea is that since the area of the cross-section can exactly be determined using definite integrals, the volume of S can also be determined by summing up all the area slices of the cross-section.

With that idea in mind, now consider calculating the volume of the solid shown below, from $x = a$ to $x = b$.

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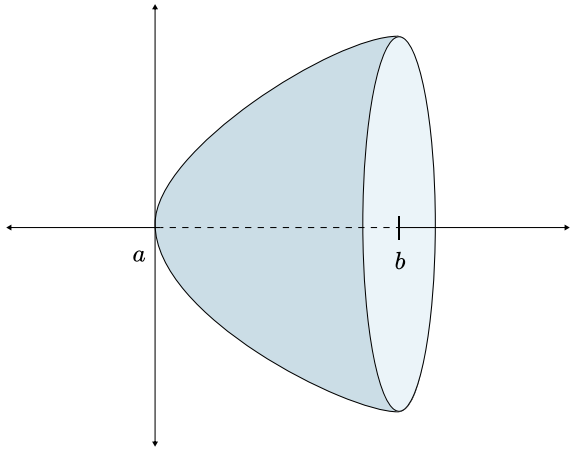

Figure 8.6

To calculate the volume of the solid within the interval $[a, b]$, the solid is intersected with a plane to form a cross-section. The area of this cross-section is represented by a known, integrable function $A(x)$. Next, the interval $[a, b]$ is divided into n sub-intervals, each with an equal thickness of $\Delta x = \dfrac{b-a}{n}$. The ith sub-interval is denoted by $[x_{i-1}, x_i]$. Within each sub-interval, we let x_i^* represent an arbitrary point, as shown below.

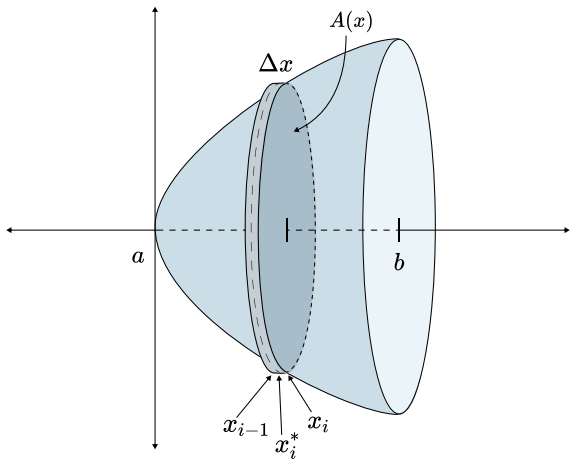

Figure 8.7

Now, let the area of the cross-section (i.e., the cross-sectional area) at the point x_i^* is $A(x_i^*)$ and thus, the volume V_i at the point x_i^* can be approximated as

$$V_i \approx A(x_i^*) \cdot \Delta x$$

Adding these approximations together we can derive an approximation for the

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total volume V over the interval $[a, b]$. That is,

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

This forms the Riemann sum. Now, as n increases indefinitely, the approximation gets better and better to the actual volume V over the interval $[a, b]$. Thus, we take the limit of the sum $\sum_{i=1}^{n} A(x_i^*)\Delta x$ as $n \to \infty$ as follows,

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x \\ &= \int_a^b A(x_i^*)dx \end{aligned}$$

As a result, we have derived a definition of the volume of a solid in terms of a definite integral.

Definition 8.3.1. (The Definition of the Volume of a Solid)
Let S be a solid enclosed between $x = a$ and $x = b$. If the cross-sectional area of S perpendicular to the x-axis at each point $x \in [a, b]$ is given by $A(x)$, then the volume V of the solid is

$$V = \int_a^b A(x)dx$$

provided that $A(x)$ is integrable.

Having defined the definition of the volume of a solid in terms of the definite integral, we now turn our attention to discuss a specific type of solids known as *solids of revolution*. These are three-dimensional shapes formed by rotating a two-dimensional region around a specified axis. Our goal in the following subsection will be to discuss the methods used to calculate the volumes of such solids systematically.

Chap. 8 / Sec. 8.3 / Subsec. 8.3.2 : The Disk Method

The Disk Method is used to calculate the volume of a solid of revolution. For that, suppose $f(x) \geq 0$ is a continuous function on the interval $[a, b]$. Let

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R represent the region enclosed by the graph of $f(x)$ and the x-axis over the interval $[a, b]$, as shown below.

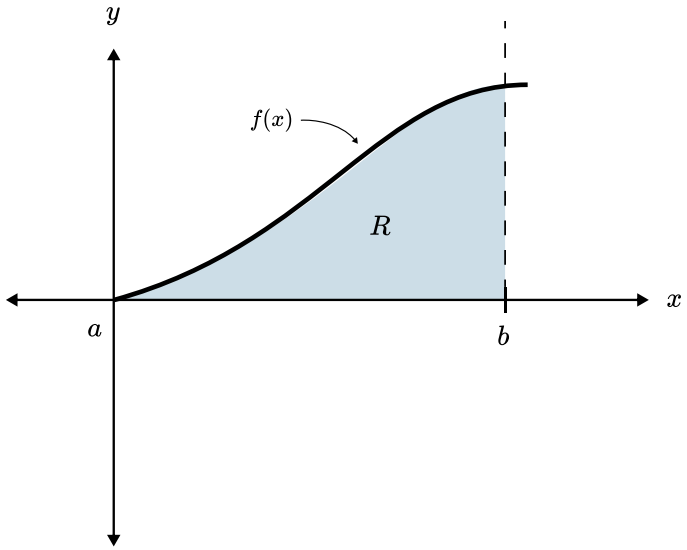

Figure 8.8

If we rotate this region R around the x-axis, it forms a three-dimensional solid known as a solid of revolution. Our goal is to find the volume of this solid, which can be done using the slicing method aforementioned.

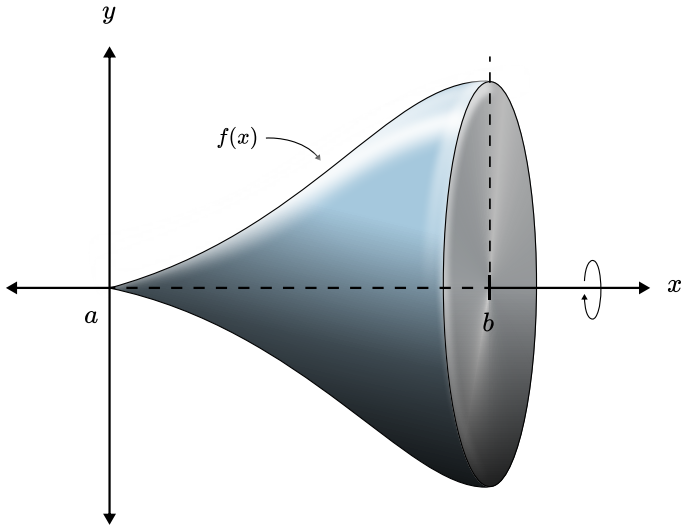

Figure 8.9

Now, in order to determine the volume of this solid of revolution over the interval $[a, b]$, we divide the interval into n sub-interval of equal thickness $\Delta x = \dfrac{b-a}{n}$.

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Then $[x_{i-1}, x_i]$ is the ith sub-interval. Let x_i^* be a point in the ith sub-interval $[x_{i-1}, x_i]$, as shown below.

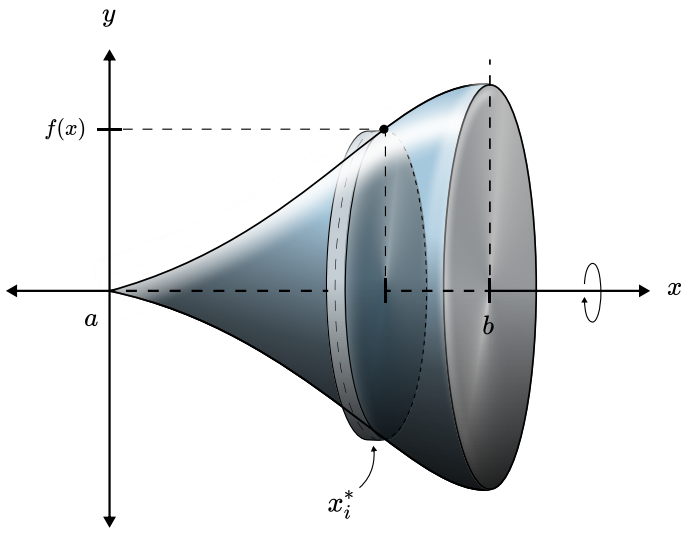

Figure 8.10

The area of the cross-section at the point x_i^* is $A(x_i^*)$. Notice that $A(x_i^*)$ has a specific form. That is, all the cross-sections perpendicular to the x-axis form a *circular disk* whose radius is the function's value $f(x)$. For this reason, the cross-section area yields,

$$\begin{aligned} A(x_i^*) &= \pi \cdot (\text{radius})^2 \\ &= \pi f(x_i^*)^2 \end{aligned}$$

Therefore, by the previously discussed slicing method, the volume of the solid yields,

$$\begin{aligned} V &= \int_a^b A\,(x)\,dx \\ &= \int_a^b \pi f(x)^2 dx \end{aligned}$$

Thus, we have the following definition.

Theorem 8.3.1. (The Disk Method)
Let $f(x)$ be a non-negative continuous function on the interval $[a, b]$. If R is the region bounded by the graph of $f(x)$ and the x-axis, over the

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interval $[a, b]$; then the volume V of the solid of revolution that is obtained by rotating R about the x-axis is the following,

$$V = \int_a^b \pi f(x)^2 dx$$

A few examples are provided below.

► **Example 8.3.1.** Let R be the region bounded by the graph of $f(x) = x^2$ and the x-axis, over the interval $[0, 5]$. What is the volume of the solid of revolution obtained by rotating R about the x-axis?

Solution. We know that the volume of a solid of revolution is,

$$V = \int_a^b \pi f(x)^2 dx$$

Where, $f(x) = x^2$ and $a = 0$ and $b = 5$. Therefore, the volume yields,

$$\begin{aligned} V &= \int_0^5 \pi (x^2)^2 dx \\ &= \pi \int_0^5 x^4 dx \\ &= \pi \left[\frac{x^5}{5} \right]_0^5 \\ &= 625\pi \end{aligned}$$

Therefore, the volume of the solid of revolution obtained by rotating R about the x-axis is 625π. ∎

► **Example 8.3.2.** Let R be the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis, over the interval $[0, 100]$. What is the volume of the solid of revolution obtained by rotating R about the x-axis?

Solution. We know that the volume of a solid of revolution is,

$$V = \int_a^b \pi f(x)^2 dx$$

Where, $f(x) = x^2$ and $a = 0$ and $b = 100$. Therefore, the volume yields,

$$V = \int_0^{100} \pi \sqrt{x} dx$$

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$$
\begin{aligned}
&= \pi \int_0^{100} x^{\frac{1}{2}} dx \\
&= \pi \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^{100} \\
&= \frac{2000\pi}{3}
\end{aligned}
$$

Therefore, the volume of the solid of revolution obtained by rotating R about the x-axis is $\frac{2000\pi}{3}$. ∎

Chap. 8 / Sec. 8.3 / Subsec. 8.3.3 : The Washer Method

A slight variation on the disk method enables us to compute the volume of more complicated solids of revolution. For example, consider R is a region bounded by the graph of $f(x)$ and $g(x)$ on the interval $[a, b]$, where $f(x) \geq g(x) \geq 0$, as shown below.

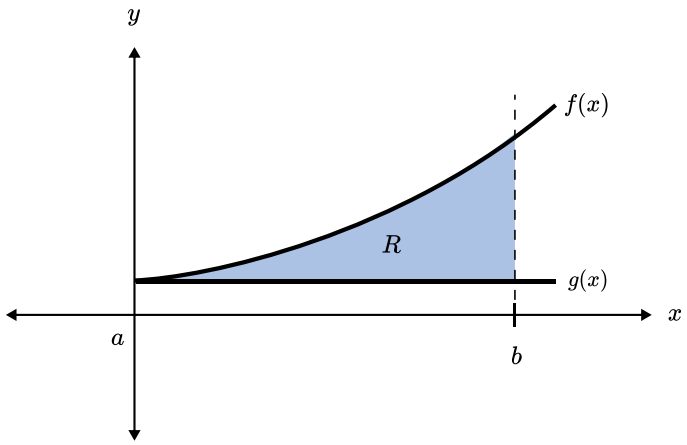

Figure 8.11

If we revolve R about the x-axis, the solid that forms has a hole through it. In other words, we get a hollowed-out solid of revolution, as shown below.

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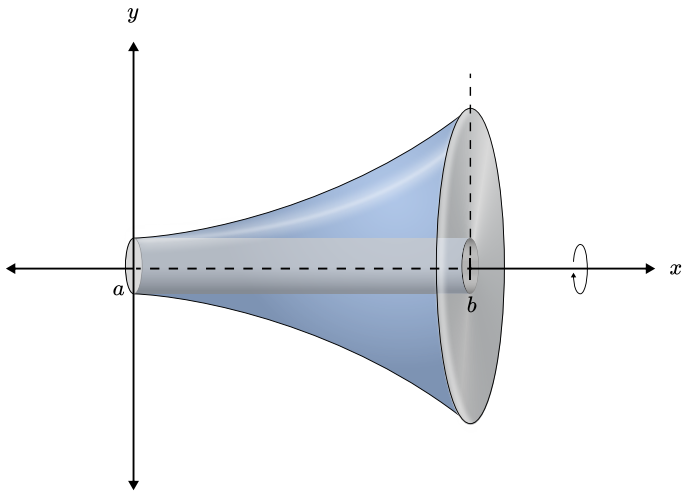

Figure 8.12

Now, in order to determine the volume of this solid of revolution over the interval $[a, b]$, we divide the interval into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$. Then $[x_{i-1}, x_i]$ is the ith sub-interval. Let x_i^* be a point in the ith sub-interval $[x_{i-1}, x_i]$, as shown below.

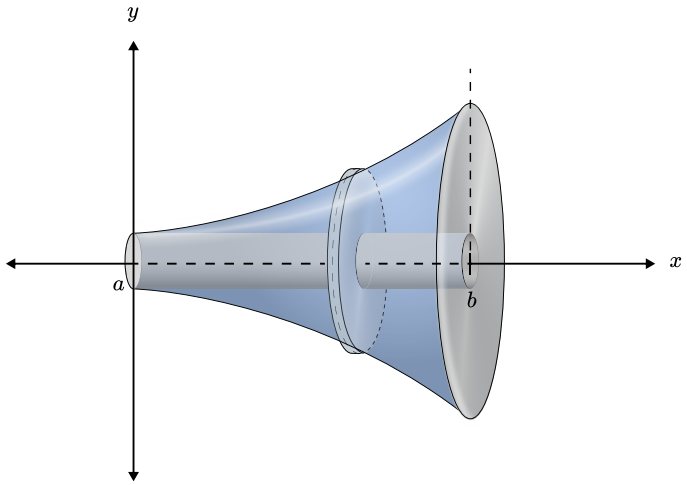

Figure 8.13

Notice that, this time, the cross-section through the solid perpendicular to the x-axis forms a circular washer (a ring-shaped solid). This circular washer has an outer radius $R = f(x)$ and an inner radius $r = g(x)$, where $a \leq x \leq b$. This is shown in more detail in the following,

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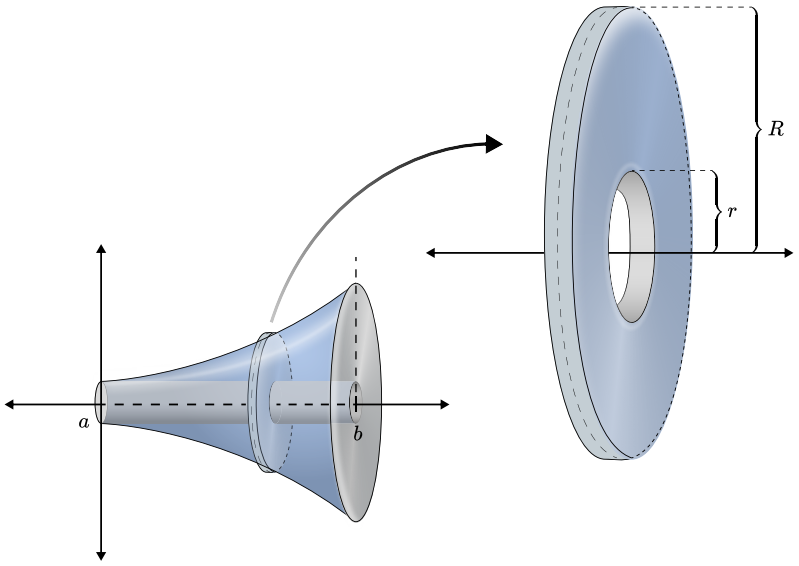

Figure 8.14

For this reason, the cross-sectional area of the circular washer is the area of the entire disk (that is, $\pi f(x)^2$) *minus* the area of the hole (that is, $\pi g(x)^2$). This implies that the area of the circular washer is as follows,

$$\begin{aligned} A(x) &= \pi R^2 - \pi r^2 \\ &= \pi f(x)^2 - \pi g(x)^2 \\ &= \pi \left(f(x)^2 - g(x)^2 \right) \end{aligned}$$

Since we've found the area of the cross-section, we can apply the slicing method to determine the volume of the solid as follows.

Theorem 8.3.2. (The Washer Method)
Let $f(x)$ and $g(x)$ be a continuous function on the interval $[a, b]$ with $f(x) \geq g(x) \geq 0$. If R is the region bounded by the graph of $y = f(x)$ and $y = g(x)$ over the interval $[a, b]$; then the volume V of the solid of revolution that is obtained by rotating R about the x-axis is the following,

$$V = \int_a^b \pi \left(f(x)^2 - g(x)^2 \right) dx$$

► **Example 8.3.3.** Suppose R is a region bounded by the graph of $f(x) = \sqrt{x}$ and $g(x) = x^2$ on the interval $[0, 1]$. What is the volume of the solid that is obtained by rotating the region R about the x-axis?

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Solution. Since $f(x) \geq g(x)$ on the interval $[0, 1]$, we'll use the washer method. The area of the cross-section at the point x is,

$$\begin{aligned} A(x) &= \pi\left(f(x)^2 - g(x)^2\right) \\ &= \pi\left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right) \\ &= \pi\left(x - x^4\right) \end{aligned}$$

By the washer method, the volume yields,

$$\begin{aligned} V &= \int_0^1 \pi\left(x - x^4\right) dx \\ &= \pi \int_0^1 \left(x - x^4\right) dx \\ &= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \\ &= \frac{3\pi}{10} \end{aligned}$$

∎

———————The End of Section 8.3
Volume: The Slicing Method

§ 8.4 Volume: The Method of Cylindrical Shells

The Method of Cylindrical Shells is another method to calculate the volume of a solid of revolution. This method is particularly useful when the solid is difficult to calculate in terms of the previously explained slicing method (whether, it is disks or washers). To demonstrate the method, let us consider R be the region bounded by the graph of $f(x)$ and the x-axis over the interval $[a, b]$, where $0 \leq a < b$ and $f(x) \geq 0$, as shown below.

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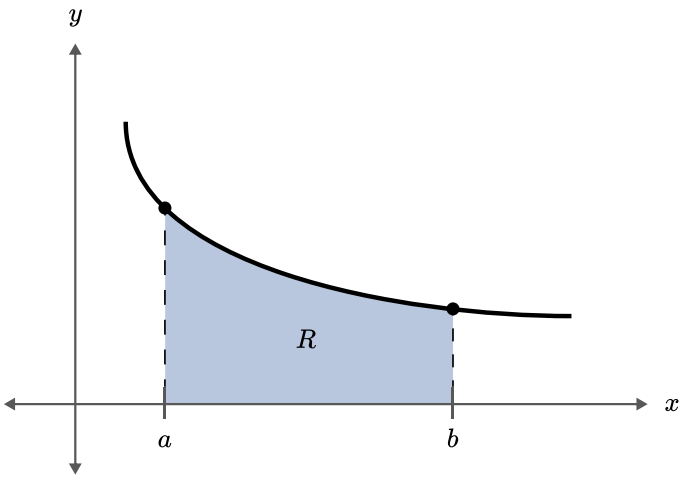

Figure 8.15

It is evident that when this region R revolves around the y-axis, a solid of revolution is generated. Our objective is to determine the volume of this solid.

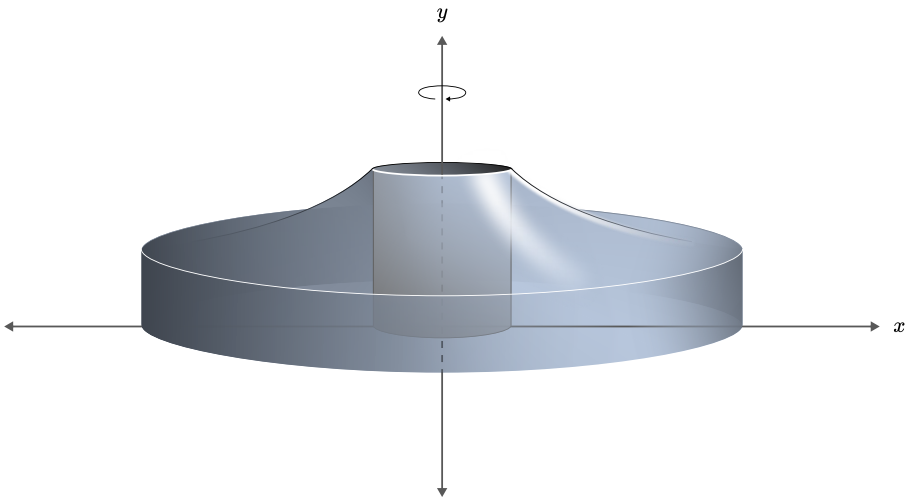

Figure 8.16

To do this, we divide the interval $[a, b]$ into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$. Then $[x_{i-1}, x_i]$ is the ith sub-interval for $i = 1, 2, 3, \cdots, n$. Let x_i^* be any arbitrary point on the ith sub-interval. Now, we build a rectangle on the ith sub-interval with a height of $f(x_i^*)$ and a width of Δx, as shown below.

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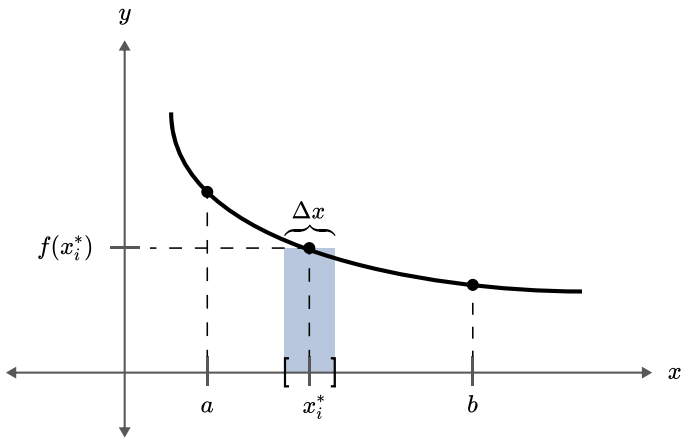

Figure 8.17

Notice that as the region R revolves around the y-axis, the rectangle built on the ith sub-interval generates a thin cylindrical shell, as shown below.

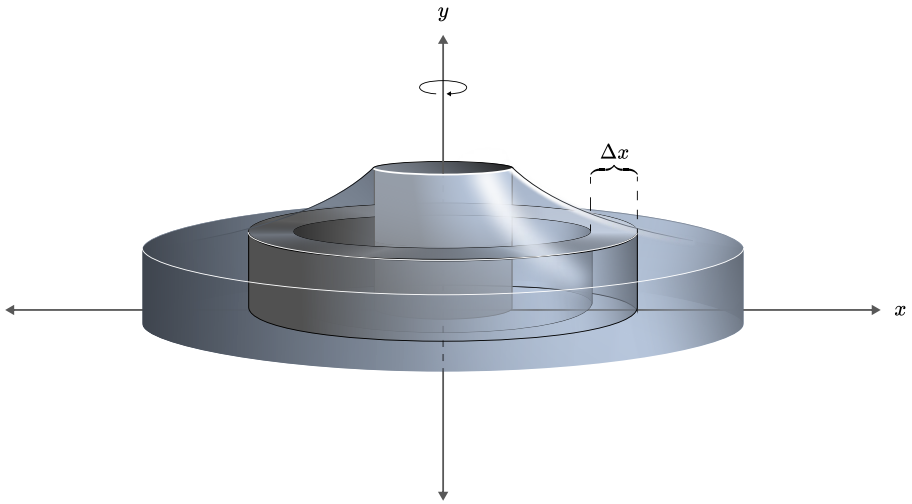

Figure 8.18

When this cylindrical shell is unwrapped, it approximately forms a thin rectangular slab. The height of this rectangular slab is the height of the original rectangle $f(x_i^*)$ and the width is also the original width of the rectangle x_i^*. The length of the rectangular slab is the circumference of the cylindrical shell's circle that has a radius of $(x_i^* - 0) = x_i^*$. Therefore, the length yields $2\pi x_i^*$. As a result, the volume of the $i$the shell is approximately,

$$\underbrace{2\pi x_i^*}_{\text{length}} \cdot \underbrace{f(x_i^*)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}} = 2\pi x_i^* f(x_i^*) \Delta x$$

Summing the volume of the n cylindrical shells gives an approximation to the

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actual volume of the entire solid of revolution. That is,

$$V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x$$

Now, as $n \to \infty$, we obtain the exact volume of the solid of revolution as a definite integral.

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x \\ &= \int_a^b 2\pi x f(x) dx \end{aligned}$$

The formulation above can be generalized for a revolving region bounded by two curves $f(x)$ and $g(x)$, where $f(x) \geq g(x)$, instead of a region bounded by the only function $f(x)$ and the x-axis on the interval $[a, b]$.

In order to do this, only the height of the rectangle built on an arbitrary point x_i^* needs to be adjusted in the previous formulation. That is, instead of height being $f(x_i^*)$ on the ith interval, the height now is the difference between two functions, that is, $(f(x_i^*) - g(x_i^*))$. So, the generalized formulation yields,

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \left(f(x_i^*) - g(x_i^*)\right) \Delta x \\ &= \int_a^b 2\pi x \left(f(x) - g(x)\right) dx \end{aligned}$$

As a result, we have the following,

Theorem 8.4.1. (The Method of Cylindrical Shells)
Let $f(x)$ and $g(x)$ be continuous functions where $f(x) \geq g(x)$ on the interval $[a, b]$. If R is the region bounded by the graph of $f(x)$ and $g(x)$, over the interval $[a, b]$; then the volume V of the solid of revolution that is obtained by rotating R about the y-axis is the following,

$$V = \int_a^b 2\pi x \Big(f(x) - g(x)\Big) dx$$

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► **Example 8.4.1.** Let R be the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis on the interval $[0, 4]$. What is the volume of the solid of revolution generated when the region R is rotated about the y-axis?

Solution. The volume of the solid of revolution can be determined by the shell method, where x is the radius and $f(x) = \sqrt{x}$ is the height of the cylindrical shell. Therefore, the volume of the solid by the shell method on the interval $[0, 4]$ yields,

$$\begin{aligned}
V &= \int_a^b 2\pi x f(x) dx \\
&= \int_0^4 2\pi x \sqrt{x} dx \\
&= 2\pi \int_0^4 x^1 x^{\frac{1}{2}} dx \\
&= 2\pi \int_0^4 x^{\frac{3}{2}} dx \\
&= 2\pi \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^4 \\
&= 2\pi \left[\frac{2x^{\frac{5}{2}}}{5} \right]_0^4 \\
&= 2\pi \left[\frac{2x^{\frac{5}{2}}}{5} \right]_0^4 \\
&= 2\pi \cdot \frac{64\pi}{5} \\
&= \frac{128\pi}{5}
\end{aligned}$$

Therefore, the volume of the solid of revolution is $\frac{128\pi}{5}$. ∎

► **Example 8.4.2.** Let R be the region bounded by the graph of $f(x) = \sin(x)$ and the x-axis on the interval $\left[0, \frac{\pi}{2}\right]$. What is the volume of the solid of revolution generated when the region R is rotated about the y-axis?

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Solution. The volume of the solid of revolution can be determined by the shell method, where x is the radius and $f(x) = \sin(x)$ is the height of the cylindrical shell. Therefore, the volume of the solid by the shell method on the interval $[0, \frac{\pi}{2}]$ is,

$$\begin{aligned} V &= \int_a^b 2\pi x f(x) dx \\ &= \int_0^{\frac{\pi}{2}} 2\pi x \sin(x) dx \\ &= 2\pi \int_0^{\frac{\pi}{2}} x \sin(x) dx \end{aligned}$$

Now, to solve the integral $\int_0^{\frac{\pi}{2}} x \sin(x) dx$, integration by parts is used as follows.

$$\begin{aligned} u = x \qquad &\Longrightarrow \qquad du = dx \\ dv = \sin(x) dx \qquad &\Longrightarrow \qquad v = \int dv = \int \sin(x) dx = -\cos(x) \end{aligned}$$

As a result, by integration by parts, we write the following.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin(x) dx &= \Big[-x\cos(x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(x) dx \\ &= \Big[-x\cos(x)\Big]_0^{\frac{\pi}{2}} + \Big[\sin(x)\Big]_0^{\frac{\pi}{2}} \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

This implies the following,

$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} x \sin(x) dx \\ &= 2\pi \cdot 1 \\ &= 2\pi \end{aligned}$$

Therefore, the volume of the solid of revolution is 2π. ∎

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The End of Section 8.4
Volume: The Method of Cylindrical Shells

§ 8.5 Arc Length

Curves, unlike straight lines, change their direction throughout their path. Therefore, calculating the precise length of a curve, such as $y = f(x)$, requires more than simply measuring the overall span from one point to another. That is, it requires us to account for every subtle bend its path.

Nevertheless, we can calculate the precise length of a curve in terms of an integral using the formula for arc length. For that, let $f(x)$ be a curve such that $f'(x)$ exists and is continuous over an interval $[a, b]$. Such a curve $f(x)$ is said to be *smooth* on $[a, b]$. Now, our objective is to determine the length of the curve from the point $(a, f(a))$ to $(b, f(b))$, as shown below.

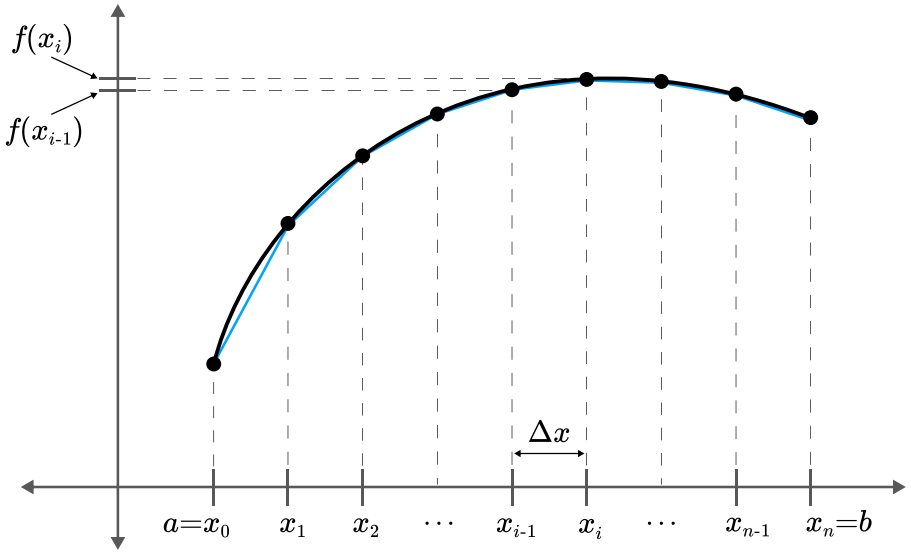

Figure 8.19

We partition the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = \dfrac{b-a}{n}$ using points $a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b$, and connect the corresponding points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), \cdots, (x_{n-1}, f(x_{n-1})), (x_n, f(x_n))$ on the graph of $f(x)$ with line segments, resulting in a polygonal line with n line segments. When n is sufficiently large (and hence Δx is small), the length of this polygonal line is a good approximation to the actual curve.

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Since we've used a regular partition, the change in the horizontal distance on every interval is fixed, that is, Δx. However, the change in the vertical distance varies from interval to interval. In general, the change in the vertical distance on the ith interval is $\Delta y_i = f(x_i) - f(x_{i-1})$, for $i = 1, 2, 3, \cdots, n$. This implies that the line segment over the ith sub-interval represents the hypotenuse of a right triangle with sides of length Δx and $|\Delta y_i| = |f(x_i) - f(x_{i-1})|$, as shown below,

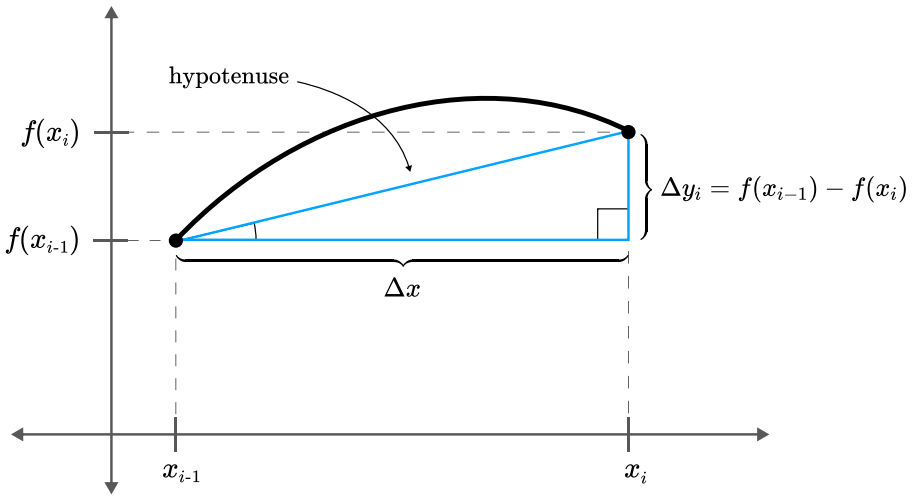

Figure 8.20

Thus, by the Pythagorean theorem,

$$\text{hypontenuse}^2 = (\Delta x)^2 + |\Delta y_i|^2$$
$$\text{hypontenuse} = \sqrt{(\Delta x)^2 + |\Delta y_i|^2}$$

Since, the hypotenuse is the ith line segment for $i = 1, 2, 3, \cdots, n$, therefore the length of each line segment is,

$$\sqrt{(\Delta x)^2 + |\Delta y_i|^2} \quad \text{for, } i = 1, 2, 3, \cdots, n$$

This can be written as,

$$\begin{aligned}\sqrt{(\Delta x)^2 + |\Delta y_i|^2} &= \sqrt{(\Delta x)^2 + (\Delta x)^2 \cdot \frac{(\Delta y_i)^2}{(\Delta x)^2}} \\ &= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2\right)}\end{aligned}$$

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$$= \sqrt{(\Delta x)^2} \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$
$$= \Delta x \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$

Now, summing these lengths together for $i = 1, 2, 3, \cdots, n$, we get the length of the polygonal lines that approximates the actual length L of the curve. that is,

$$L \approx \sum_{i=1}^{n} \Delta x \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$

Notice that $\dfrac{\Delta y_i}{\Delta x}$ represents the slope of the line segment over the ith interval. Now, by the Mean Value Theorem, this slope equals the derivative $f'(x_i^*)$ for some point $x_i^* \in [x_{i-1}, x_i]$. Therefore,

$$L \approx \sum_{i=1}^{n} \Delta x \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$
$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \cdot \Delta x$$
$$= \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2} \Delta x$$

This forms the Riemann sum. Now, letting $n \to \infty$, the sum approaches a definite integral which is the exact length L of the curve. That is,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2} \Delta x$$
$$= \int_a^b \sqrt{1 + f'(x)^2} dx$$

As a result, we have the following definition for the arc length of $y = f(x)$.

Definition 8.5.1. (Arc Length of a Function) Let $f(x)$ be a function such that $f'(x)$ exists and is continuous over an interval $[a, b]$, then the length L of the curve $f(x)$ from the point $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

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A few examples are provided below to demonstrate the definition above.

► **Example 8.5.1.** Find the length of the graph of $f(x) = 5x + 3$ from $x = 0$ to $x = 5$.

Solution. Since $f(x) = 5x+3$, therefore, $f'(x) = 5$. Thus, we write the following,

$$\begin{aligned} L &= \int_0^5 \sqrt{1 + 5^2} dx \\ &= \sqrt{26} \cdot \int_0^5 dx \\ &= \sqrt{26} \cdot x \Big]_0^5 \\ &= \sqrt{26} \cdot 5 \\ &\approx 25.495 \end{aligned}$$

Therefore, the length of the graph of $f(x) = 5x + 3$ from $x = 0$ to $x = 5$ is approximately 25.495. ∎

► **Example 8.5.2.** Find the length of the graph of $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = 1$.

Solution. Since $f(x) = \ln(\cos(x))$, therefore, $f'(x) = -\dfrac{\sin(x)}{\cos(x)}$. Thus, we write the following,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(-\frac{\sin(x)}{\cos(x)}\right)^2} dx \\ &= \int_0^1 \sqrt{1 + (-\tan(x))^2} dx \\ &= \int_0^1 \sqrt{1 + \tan^2(x)} dx \\ &= \int_0^1 \sqrt{\sec^2(x)} dx \end{aligned}$$

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$$
\begin{aligned}
&= \int_0^1 \sec(x)dx \\
&= \ln|\sec(x) + \tan(x)|\Big]_0^1 \\
&= \ln|\sec(1) + \tan(1)| - \ln|\sec(0) + \tan(0)| \\
&\approx 1.23
\end{aligned}
$$

Therefore, the length of the graph of $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = 1$ is approximately 1.23. ∎

———————The End of Section 8.5
Arc Length

§ 8.6 The Area of a Surface of Revolution

In the previous section, we discussed solids of revolution and explored various methods to calculate their volumes. Now, we turn our attention to calculate the area of the surface of such solids of revolution. However, before diving into this topic, we need to first understand the formula for the lateral surface area of a frustum of a right circular cone, which in turn requires us to know the formula for the lateral surface area of a right circular cone.

It is known from geometry that the lateral surface area A of the following right circular cone

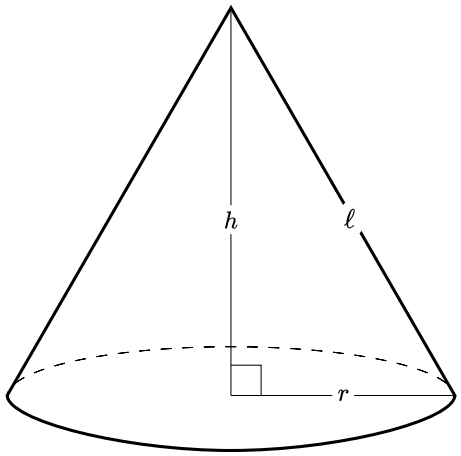

Figure 8.21

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is given by the following formula:

$$A = \pi r \ell$$

where, r is the radius of the base of the cone and ℓ is the slant height of the cone, which is equal to $\sqrt{r^2 + h^2}$ by the Pythagorean theorem. As a result, we have the formula for the lateral surface area A of a right circular cone as,

$$\begin{aligned} A &= \pi r \ell \\ &= \pi r \sqrt{r^2 + h^2} \end{aligned}$$

Now, using the formula for the lateral surface area of a right circular cone, we will derive the formula for the lateral surface area of a *frustum* of a right circular cone. A frustum of a right circular cone is a part of a cone that remains after the top portion of the cone is cut off by a plane parallel to the base.

In order to derive the formula for the frustum, consider a linear function $f(x) = nx$ over an interval $[a, b]$, where $0 < a < b$ and $n > 0$, as shown below.

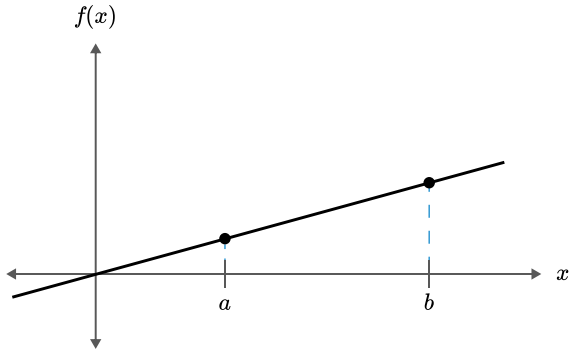

Figure 8.22

Now, rotate this line segment about the x-axis resulting in a frustum of a cone as shown below.

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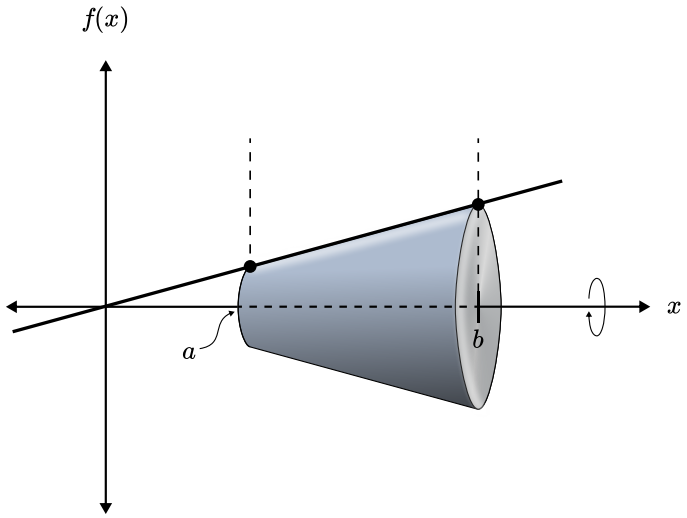

Figure 8.23

We need to determine the surface area S of this frustum. The idea is to subtract the small surface area of the frustum over $[0, a]$ from the larger surface area of the frustum over $[0, b]$. This allows us to calculate the surface area of the frustum over $[a, b]$.

Now, notice carefully that the radius r of the cone over $[0, b]$ is $f(b) = nb$ and the height h over $[0, b]$ is b. Thus, the surface area of the cone over $[0, b]$ yields,

$$\begin{aligned} S_{[0,b]} &= \pi r \sqrt{r^2 + h^2} \\ &= \pi (nb) \sqrt{(nb)^2 + b^2} \\ &= \pi (nb) \sqrt{b^2 (n^2 + 1)} \\ &= \pi (nb) \sqrt{b^2} \cdot \sqrt{(n^2 + 1)} \\ &= \pi (nb) b \cdot \sqrt{(n^2 + 1)} \\ &= \pi b^2 n \sqrt{n^2 + 1} \end{aligned}$$

Similarly, the radius r of the cone over $[0, a]$ is $f(a) = na$ and the height h over $[0, a]$ is a. Thus, the surface area of the cone $[0, a]$ yields,

$$\begin{aligned} S_{[0,a]} &= \pi r \sqrt{r^2 + h^2} \\ &= \pi (na) \sqrt{(na)^2 + a^2} \\ &= \pi (na) \sqrt{a^2 (n^2 + 1)} \\ &= \pi (na) \sqrt{a^2} \cdot \sqrt{(n^2 + 1)} \\ &= \pi (na) a \cdot \sqrt{(n^2 + 1)} \end{aligned}$$

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$$= \pi a^2 n\sqrt{n^2+1}$$

Now, the surface area S of the frustum over $[a,b]$ yields the following,

$$\begin{aligned} S &= S_{[0,b]} - S_{[0,a]} \\ &= \left(\pi b^2 n\sqrt{n^2+1}\right) - \left(\pi a^2 n\sqrt{n^2+1}\right) \\ &= \pi n\left(b^2-a^2\right)\sqrt{n^2+1} \end{aligned}$$

Now, we calculate the line segment ℓ between $(a, f(a))$ and $(b, f(b))$ for further simplification in the equation above. For that, note that the line segment between the point $(a, f(a)) = (a, an)$ and $(b, f(b)) = (b, bn)$ yields the following.

$$\begin{aligned} \ell &= \sqrt{(b-a)^2 + (bn-an)^2} \\ &= \sqrt{b^2 - 2ab + a^2 + (bn)^2 - 2anbn + (an)^2} \\ &= \sqrt{b^2 - 2ab + a^2 + b^2n^2 - 2abn^2 + a^2n^2} \\ &= \sqrt{b^2 - 2ab + a^2 + n^2\left(b^2 - 2ab + a^2\right)} \\ &= \sqrt{\left(b^2 - 2ab + a^2\right)\left(1+n^2\right)} \\ &= \sqrt{\left(b^2 - 2ab + a^2\right)} \cdot \sqrt{\left(1+n^2\right)} \\ &= \sqrt{(b-a)^2} \cdot \sqrt{\left(1+n^2\right)} \\ &= (b-a)\sqrt{(n^2+1)} \end{aligned}$$

As a result, we have line segment $\ell = (b-a)\sqrt{(n^2+1)}$

Now, since the surface area S of the frustum over $[a,b]$ is $\pi n\left(b^2-a^2\right)\sqrt{n^2+1}$. Therefore,

$$\begin{aligned} S &= \pi n\left(b^2-a^2\right)\sqrt{n^2+1} \\ &= \pi n \underbrace{(b+a)(b-a)}_{(b^2-a^2)}\sqrt{n^2+1} \\ &= \pi n (b+a) \underbrace{(b-a)\sqrt{n^2+1}}_{\ell} \\ &= \pi n (b+a)\,\ell \\ &= \pi\left(\underbrace{bn}_{f(b)} + \underbrace{an}_{f(a)}\right)\ell \\ &= \pi\Big(f(b) + f(a)\Big)\ell \end{aligned}$$

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Thus, we have derived the formula for the lateral surface area S of a frustum of a right circular cone that was obtained by rotating the function $f(x)$ over the interval $[a, b]$. With this result, we are now equipped to derive a general formula for the surface area of a surface of revolution.

Chap. 8 / Sec. 8.6 / Subsec. 8.6.1 : The Generalized Area Formula for a Surface of Revolution

The formula for the surface area S of a frustum of a cone in the previous subsection was found to be the following,

$$S = \pi\Big(f\,(b) + f\,(a)\Big)\ell$$

Based on this result, we now derive the general formula for the surface area of a surface of revolution. In order to do that, consider a continuous function $f(x)$ over an interval $[a, b]$, as shown below.

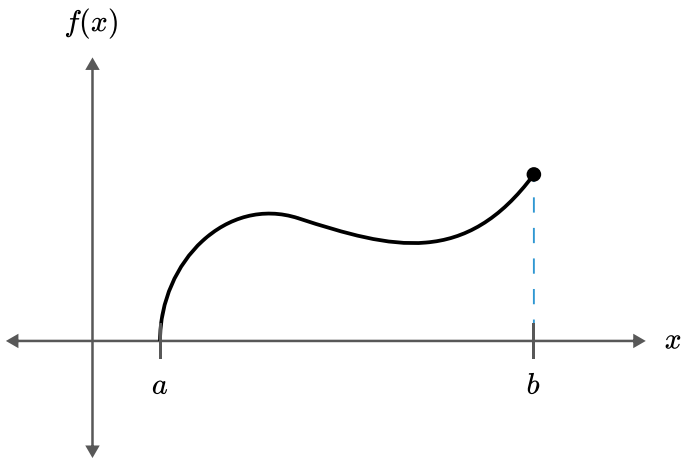

Figure 8.24

Now, rotate this curve of $f(x)$ about the x-axis, resulting in the following solid of revolution,

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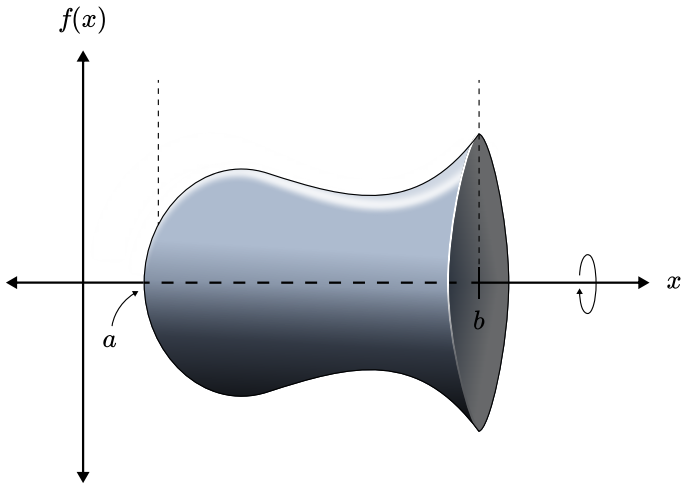

Figure 8.25

Now, we need to determine the surface area S of this solid of revolution. For that, we divide the interval $[a, b]$ into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$. Then $[x_{i-1}, x_i]$ becomes the ith sub-interval, as shown below.

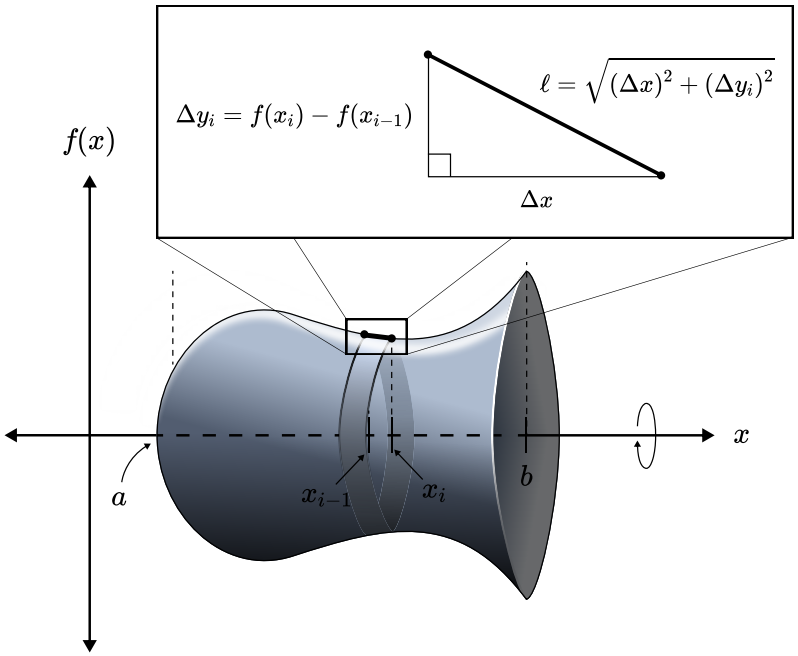

Figure 8.26

Now, notice that the line segment ℓ over the ith sub-interval, that is, the line segment between the points $(x_i, f(x_{i-1}))$ and $(x_i, f(x_i))$ is calculated as follows,

$$\begin{aligned}\ell &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}\end{aligned}$$

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Here, since all the sub-intervals have equal width, therefore width Δx is the same for all i and thus Δx doesn't have any subscripts. Notice, however, that this is not the case for the change in y, that is, $\Delta y_i = f(x_i) - f(x_{i-1})$.

As a result, the surface area of this surface of revolution over $[x_{i-1}, x_i]$ is given by the formula of the surface area of a frustrum of a cone over $[x_{i-1}, x_i]$, that is,

$$\begin{aligned} S_i &= \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \ell \\ &= \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(\Delta y_i\right)^2} \end{aligned}$$

Now, summing these individual frustums for $i = 1, 2, 3, \cdots, n$ yields the total surface area of the surface of revolution S,

$$\begin{aligned} S &\approx \sum_{i=1}^{n} S_i \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(\Delta y_i\right)^2} \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(f\left(x_i\right) - f\left(x_{i-1}\right)\right)^2} \end{aligned} \tag{1}$$

It is known—from the mean value theorem—that,

$$\begin{aligned} \frac{f\left(x_i\right) - f\left(x_{i-1}\right)}{\Delta x} &= f'\left(x_i^*\right) \\ f\left(x_i\right) - f\left(x_{i-1}\right) &= f'\left(x_i^*\right) \Delta x \end{aligned} \tag{2}$$

Now, substituting equ. (2) into (1), yields,

$$\begin{aligned} \sum_{i=1}^{n} S_i &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(f\left(x_i\right) - f\left(x_{i-1}\right)\right)^2} \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(f'\left(x_i^*\right) \Delta x\right)^2} \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 + \left(\Delta x\right)^2 f'(x_i^*)^2} \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(\Delta x\right)^2 \left(1 + f'(x_i^*)^2\right)} \\ &= \sum_{i=1}^{n} \pi \left(f\left(x_i\right) + f\left(x_{i-1}\right) \right) \sqrt{\left(1 + f'(x_i^*)^2\right)} \Delta x \end{aligned}$$

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Now, when n (the number of sub-intervals within the interval $[a, b]$) is sufficiently large, then $x_{i-1} \approx x_i \approx x_i^*$. This implies that $f(x_{i-1}) \approx f(x_i) \approx f(x_i^*)$, for $i = 1, 2, 3, \cdots, n$, given that the function $f(x)$ is continuous. Therefore,

$$\begin{aligned} S &\approx \sum_{i=1}^{n} \pi \left(f\left(x_i^*\right) + f\left(x_i^*\right)\right) \sqrt{\left(1 + f'(x_i^*)^2\right)} \Delta x \\ &= \sum_{i=1}^{n} 2\pi f\left(x_i^*\right) \sqrt{\left(1 + f'(x_i^*)^2\right)} \Delta x \end{aligned}$$

This forms the Riemann sum. Now if $n \to \infty$, then $\Delta x \to 0$, and we have the following,

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f\left(x_i^*\right) \sqrt{\left(1 + f'(x_i^*)^2\right)} \Delta x \\ &= \int_a^b 2\pi f\left(x\right) \sqrt{\left(1 + f'(x)^2\right)} dx \end{aligned}$$

As a result, we have the following definition of the area of a surface of revolution.

Definition 8.6.1. (Area of a Surface of Revolution)
Let $f(x)$ be a differentiable and non-negative function over an interval $[a, b]$. The area of the surface S formed by rotating $f(x)$ over the interval $[a, b]$ about the x-axis is given by

$$S = \int_a^b 2\pi f\left(x\right) \sqrt{\left(1 + f'(x)^2\right)} dx$$

A few examples to demonstrate the definition above are provided below.

► **Example 8.6.1.** Find the surface area generated by rotating the function $f(x) = 2$ about the x-axis over the interval $[0, 3]$.

Solution. Since $f(x) = 2$, therefore $f'(x) = 0$. As a result, we write the following,

$$\begin{aligned} S &= \int_0^3 2\pi f\left(x\right) \sqrt{\left(1 + f'(x)^2\right)} dx \\ &= \int_0^3 2\pi \cdot 2 \cdot \sqrt{(1+0)} dx \end{aligned}$$

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$$
\begin{aligned}
&= \int_0^3 4\pi dx \\
&= 4\pi \cdot \int_0^3 dx \\
&= 4\pi \cdot x \Big]_0^3 \\
&= 4\pi \cdot 3 \\
&= 12\pi
\end{aligned}
$$

Therefore, the surface area is 12π. ∎

► **Example 8.6.2.** If the graph of $f(x) = \sqrt{x}$ is rotated about the x-axis over the interval $[0, 2]$, then what is the area of the surface of this revolution?

Solution. Since $f(x) = \sqrt{x}$, therefore $f'(x) = \dfrac{1}{2\sqrt{x}}$. As a result, we write the following,

$$
\begin{aligned}
S &= \int_0^2 2\pi \left(\sqrt{x}\right) \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \\
&= 2\pi \int_0^2 \sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \\
&= 2\pi \int_0^2 \sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}} dx \\
&= 2\pi \int_0^2 \sqrt{x} \cdot \sqrt{\frac{4x+1}{4x}} dx \\
&= 2\pi \int_0^2 \cancel{\sqrt{x}} \cdot \frac{\sqrt{4x+1}}{2\cancel{\sqrt{x}}} dx \\
&= 2\pi \int_0^2 \frac{\sqrt{4x+1}}{2} dx \\
&= \frac{1}{2} 2\pi \int_0^2 \sqrt{4x+1} dx
\end{aligned}
$$

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$$= \pi \cdot \int_0^2 (4x+1)^{\frac{1}{2}} dx \tag{1}$$

Now, in order to solve the integral $\pi \cdot \int_0^2 (4x+1)^{\frac{1}{2}} dx$, we use the substitution rule (that is, the u-substitution). For that, let $u = 4x+1$, then

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}(4x+1) \\ &= 4 \\ \therefore dx &= \frac{du}{4} \end{aligned}$$

Now, for the limit of integration, we do the following,

$$\begin{aligned} &\text{For} \quad x = 0 \quad \Longrightarrow \quad u = 4 \cdot 0 + 1 = 1 \\ &\text{For} \quad x = 2 \quad \Longrightarrow \quad u = 4 \cdot 2 + 1 = 9 \end{aligned}$$

Now, substituting $u = 4x+1$, $dx = \dfrac{du}{4}$, and the limits of integration into the equation (1), we get the following,

$$\begin{aligned} \pi \cdot \int_0^2 (4x+1)^{\frac{1}{2}} dx &= \pi \cdot \int_1^9 u^{\frac{1}{2}} \frac{du}{4} \\ &= \frac{\pi}{4} \cdot \int_1^9 u^{\frac{1}{2}} \, du \\ &= \frac{\pi}{4} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Bigg]_1^9 \\ &= \frac{\pi}{4} \cdot \left(\frac{9^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{3}{2}}}{\frac{3}{2}} \right) \\ &= \frac{\pi}{4} \cdot \frac{52}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

Therefore, the surface area is $\dfrac{13\pi}{3}$. ∎

The End of Section 8.6

The Area of a Surface of Revolution

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§ 8.7 Work

In everyday language, we often use the word "work" to mean any kind of task or activity we do, whether it involves any sort of movement. For example, when we say, "I did a lot of work today" we typically emphasize the fact that a lot of energy or effort has been used to get something done.

However, this common understanding of work doesn't align with the scientific definition of work. For example, simply holding a heavy object in place requires energy and effort, but in the scientific sense, no work is done if there is no movement. Mathematically, work is defined as a measurable quantity that only occurs when a force F moves an object in the direction of that force. That is, work is defined as follows.

$$W = F \cdot d \tag{1}$$

where, $W =$ work, $F =$ force, and $d =$ displacement.

In defining work, the equation above only works when the force remains constant throughout its entire application. But in reality, the application of force often changes. For example, imagine we're pushing an object along a path, and the force we exert changes from point to point as we move along. Let's say we're moving the object from $x = a$ to $x = b$ along the x-axis by a continuously varying force $F(x)$, as shown in the figure below.

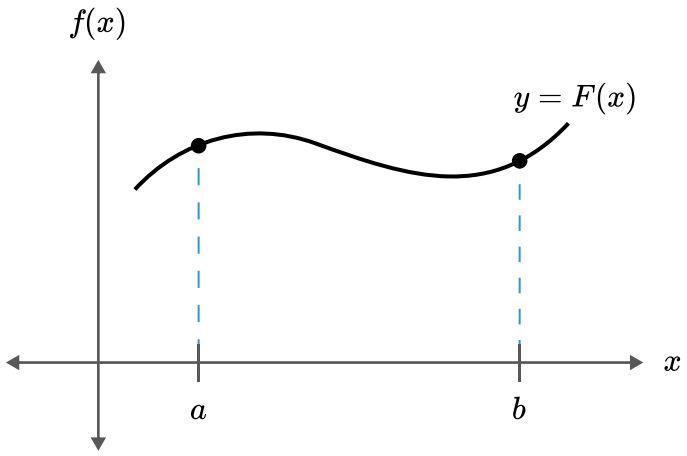

Figure 8.27

Now the question is how much work is done in moving that object based on the definition of work we have seen so far? Equation (1) cannot be directly

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applied to this problem because it assumes a constant force. However, by the help of integration, we can adapt Equation (1) to solve the problem even when the force is not constant. The idea lies in the realization that the area of the region bounded by the graph of $y = F(x)$ and the x-axis over the interval $[a, b]$ represents the work done by the force $F(x)$ over that distance, as shown below.

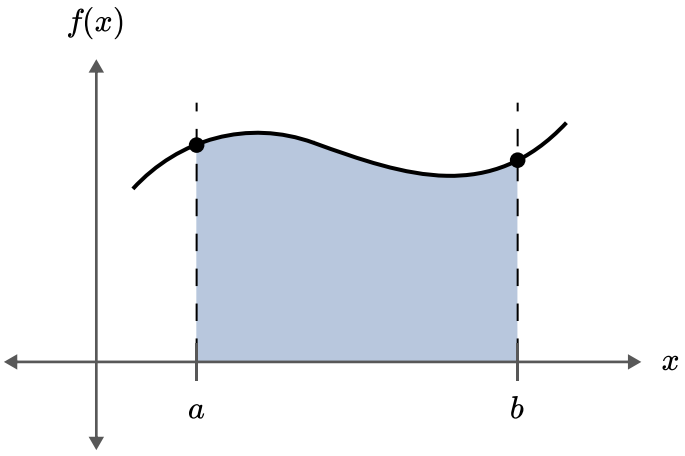

Figure 8.28

Now, if we calculate this area under the graph, we can determine the work done by the force $F(x)$ over that interval. To calculate the area under the graph, we divide the interval $[a, b]$ into n sub-intervals that are sufficiently small such that the force does not change much, that is, remains nearly constant within each sub-divided sub-interval. This approach allows us to treat the force as constant over each sub-interval, making it easier to approximate the work done on each sub-interval using the formula for work. By adding together these small approximations for each sub-interval, we create a Riemann sum. This Riemann sum gives us an approximation of the total work W done over the entire interval $[a, b]$. Finally, we take the limit of these Riemann sums as the number of sub-intervals increases without a bound, and thus obtain the actual work W done in moving the object.

In order to do this, let us divide the interval $[a, b]$ into n sub-interval of equal width $\Delta x = \dfrac{b-a}{n}$ by using points $a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b$ between $x = a$ and $x = b$. Let x_i^* be any point in the ith sub-interval, for $i = 1, 2, 3, \cdots, n$. Since, the interval $[a, b]$ is divided into sub-intervals in a way that the force within each sub-interval remains approximately constant, therefore the force in the ith sub-interval is roughly constant with a value $F(x_i^*)$. Thus, the work done in moving the object across the ith sub-interval is approximately $F(x_i^*) \cdot \Delta x$,

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according to Equation (1). Similarly, summing the work done over every sub-interval, we get an approximation of the total work W done over the interval $[a, b]$, that is,

$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x$$

Now, if we let the number of sub-interval $n \to \infty$, then $\sum_{i=1}^{n} F(x_i^*)\Delta x \to \int_a^b F(x)dx = W$. Thus, taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x \\ &= \int_a^b F(x)dx \end{aligned}$$

As a result, we have the following definition of work done by a variable force.

Definition 8.7.1. (Work Done by a Variable Force) If an object is moved by a varying force F from $x = a$ to $x = b$ in the direction of the x-axis, then the work W done in moving the object is

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x \\ &= \int_a^b F(x)dx \end{aligned}$$

A few examples to demonstrate the definition above are shown below.

► **Example 8.7.1.** Find the work done by a force $F(x) = e^x$ applied over the distance from $x = 0$ to $x = 1$.

Solution. Using the definition of work, we write the following,

$$\begin{aligned} \int_0^1 e^x dx &= e^x \Big]_0^1 \\ &= e^1 - e^0 \\ &\approx 1.718 \end{aligned}$$

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Therefore, the work done by a force $F(x) = e^x$ applied over the distance from $x = 0$ to $x = 1$ is approximately 1.718 units. ∎

► **Example 8.7.2.** A spring has a spring constant $k = 200\frac{\text{N}}{\text{m}}$. How much work is required to stretch the spring from its natural length to a displacement of $x = 0.1\text{m}$?

Solution. To calculate the work done in stretching the spring, we integrate the force $F(x)$ over the displacement of 0.1m. That is,

$$W = \int_0^{0.1} F(x)dx$$

From Hooke's Law, we know that

$$F(x) = kx$$

Since, $k = 200\frac{\text{N}}{\text{m}}$, therefore,

$$\begin{aligned} F(x) &= kx \\ &= 200x \end{aligned}$$

This implies that

$$\begin{aligned} W &= \int_0^{0.1} F(x)dx \\ &= \int_0^{0.1} 200xdx \\ &= 200 \cdot \int_0^{0.1} xdx \\ &= 200 \cdot \frac{x^2}{2}\Big]_0^{0.1} \\ &= 200 \cdot \frac{0.1^2}{2} \\ &= 1\text{J} \end{aligned}$$

Therefore, the work required to stretch the spring is 1J. ∎

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Relative/Local Extreme Value

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